Quantum deformation of Whittaker modules and Toda lattice

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February 1, 2008

Abstract

In 1978 Kostant suggested the Whittaker model of the center of the universal enveloping algebra $U(\mathfrak{g})$ of a complex simple Lie algebra $\mathfrak{g}$. The main result is that the center of $U(\mathfrak{g})$ is isomorphic to a commutative subalgebra in $U(\mathfrak{b}_+)$, where $\mathfrak{b}_+$ is a Borel subalgebra in $\mathfrak{g}$. This observation is used in the theory of principal series representations of the corresponding Lie group G and in the proof of complete integrability of the quantum Toda lattice. In this paper we generalize the Kostant's construction to quantum groups. In our construction we use quantum analogues of regular nilpotent elements defined in [13]. Using the Whittaker model of the center of 5the algebra $U_h(\mathfrak{g})$ we define quantum deformations of Whittaker modules. The new Whittaker model is also applied to the deformed quantum Toda lattice recently studied by Etingof in [6]. We give new proofs of his results which resemble the original Kostant's proofs for the quantum Toda lattice.

Introduction

In 1978 Kostant suggested the *Whittaker model* of the center of the universal enveloping algebra $U(\mathfrak{g})$ of a complex simple Lie algebra $\mathfrak{g}$. An essential role in this construction is played by a non–singular character χ of the maximal

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nilpotent subalgebra $\mathfrak{n}_+ \subset \mathfrak{g}$. The main result is that the center of $U(\mathfrak{g})$ is isomorphic to a commutative subalgebra in $U(\mathfrak{b}_-)$, where $\mathfrak{b}_- \subset \mathfrak{g}$ is the opposite Borel subalgebra. This observation is used in the theory of principal series representations of the corresponding Lie group G and in the proof of complete integrability of the quantum Toda lattice.

The goal of this paper is to generalize the Kostant's construction to quantum groups. An obvious obstruction is the fact that the subalgebra in $U_h(\mathfrak{g})$ generated by positive root generators (subject to the quantum Serre relations) does not have non-singular characters. In order to overcome this difficulty we use a family of new realizations of quantum groups introduced in [13]. The modified quantum Serre relations allow for non–singular characters, and we are able to construct the Whittaker model of the center of $U_h(\mathfrak{g})$.

Using the Whittaker model of the center of $U_h(\mathfrak{g})$ we introduce quantum deformations of Whittaker modules.

The new Whittaker model is also applied to the deformed quantum Toda lattice recently studied by Etingof (see [6]). We give new proofs of his results which resemble the original Kostant's proofs for the quantum Toda lattice.

The paper is organized as follows. Section 1 contains a review of Kostant's results on the Whittaker model and Whittaker modules [10], [11]. In order to create a pattern for proofs in the quantum group case we recall most of the Kostant's proofs. The central part of the paper is Section 2. There we discuss properties of new realizations of finite-dimensional quantum groups and present the Whittaker model of the center of $U_h(\mathfrak{g})$. In Section 2.4 we introduce quantum deformed Whittaker modules. Section 2.6 contains a discussion of the deformed quantum Toda lattice.

1 Whittaker model

In this section we recall the Whittaker model of the center of the universal enveloping algebra $U(\mathfrak{g})$, where $\mathfrak{g}$ is a complex simple Lie algebra.

1.1 Notation

Fix the notation used throughout of the text. Let G be a connected simply connected finite–dimensional complex simple Lie group, $\mathfrak{g}$ its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. Choose

an ordering in the root system. Let $\alpha_i,\ i=1,\dots l,\quad l=rank(\mathfrak{g})$ be the simple roots, $\Delta_+=\{\beta_1,\dots,\beta_N\}$ the set of positive roots. Denote by ρ a half of the sum of positive roots, $\rho=\frac{1}{2}\sum_{i=1}^N\beta_i$. Let $H_1,\dots,H_l$ be the set of simple root generators of $\mathfrak{h}$.

Let a_{ij} be the corresponding Cartan matrix. Let $d_1,\dots,d_l$ be coprime positive integers such that the matrix $b_{ij}=d_ia_{ij}$ is symmetric. There exists a unique non–degenerate invariant symmetric bilinear form $(,)$ on $\mathfrak{g}$ such that $(H_i,H_j)=d_j^{-1}a_{ij}$. It induces an isomorphism of vector spaces $\mathfrak{h}\simeq\mathfrak{h}^*$ under which $\alpha_i\in\mathfrak{h}^*$ corresponds to $d_iH_i\in\mathfrak{h}$. We denote by $\alpha^\vee$ the element of $\mathfrak{h}$ that corresponds to $\alpha\in\mathfrak{h}^*$ under this isomorphism. The induced bilinear form on $\mathfrak{h}^*$ is given by $(\alpha_i,\alpha_j)=b_{ij}$.

Let W be the Weyl group of the root system Δ. W is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections $s_1,\dots,s_l$,

$$s_i(h)=h-\alpha_i(h)H_i,\quad h\in\mathfrak{h}.$$

The action of W preserves the bilinear form $(,)$ on $\mathfrak{h}$. We denote a representative of $w\in W$ in G by the same letter. For $w\in W,g\in G$ we write $w(g)=wgw^{-1}$.

Let $\mathfrak{b}_+$ be the positive Borel subalgebra and $\mathfrak{b}_-$ the opposite Borel subalgebra; let $\mathfrak{n}_+=[\mathfrak{b}_+,\mathfrak{b}_+]$ and $\mathfrak{n}_-=[\mathfrak{b}_-,\mathfrak{b}_-]$ be their nil-radicals. Let $H=\exp\mathfrak{h},N_+=\exp\mathfrak{n}_+,N_-=\exp\mathfrak{n}_-,B_+=HN_+,B_-=HN_-$ be the Cartan subgroup, the maximal unipotent subgroups and the Borel subgroups of G which correspond to the Lie subalgebras $\mathfrak{h},\mathfrak{n}_+,\mathfrak{n}_-,\mathfrak{b}_+$ and $\mathfrak{b}_-$, respectively.

We identify $\mathfrak{g}$ and its dual by means of the canonical invariant bilinear form. Then the coadjoint action of G on $\mathfrak{g}^*$ is naturally identified with the adjoint one. We also identify ${\mathfrak{n}_+}^*\cong\mathfrak{n}_-,\ {\mathfrak{b}_+}^*\cong\mathfrak{b}_-$.

Let $\mathfrak{g}_\beta$ be the root subspace corresponding to a root $\beta\in\Delta$, $\mathfrak{g}_\beta=\{x\in\mathfrak{g}|[h,x]=\beta(h)x$ for every $h\in\mathfrak{h}\}$. $\mathfrak{g}_\beta\subset\mathfrak{g}$ is a one–dimensional subspace. It is well–known that for $\alpha\neq-\beta$ the root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the canonical invariant bilinear form. Moreover $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are non–degenerately paired by this form.

Root vectors $X_\alpha\in\mathfrak{g}_\alpha$ satisfy the following relations:

$$[X_\alpha,X_{-\alpha}]=(X_\alpha,X_{-\alpha})\alpha^\vee.$$

1.2 The Whittaker model

In this section we introduce the Whittaker model of the center of the universal enveloping algebra $U(\mathfrak{g})$. We start by recalling the classical result of Chevalley which describes the structure of the center.

Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The standard filtration $U_k(\mathfrak{g})$ in $U(\mathfrak{g})$ induces a filtration $Z_k(\mathfrak{g})$ in $Z(\mathfrak{g})$. The following important theorem may be found for instance in [1], Ch.8, §8, no. 3, Corollary 1 and no.5, Theorem 2.

Theorem (Chevalley) *One can choose elements $I_k \in Z_{m_k+1}(\mathfrak{g}), \quad k = 1, \ldots l$, where m_k are called the exponents of $\mathfrak{g}$, such that $Z(\mathfrak{g}) = \mathbb{C}[I_1, \ldots, I_l]$ is a polynomial algebra in l generators.*

The adjoint action of G on $\mathfrak{g}$ naturally extends to $S(\mathfrak{g})$. Let $S(\mathfrak{g})^G$ be the algebra of G–invariants in $S(\mathfrak{g})$. Clearly, $Gr Z(\mathfrak{g}) \cong S(\mathfrak{g})^G$. In particular $S(\mathfrak{g})^G \cong \mathbb{C}[\widehat{I}_1, \ldots, \widehat{I}_l]$, where $\widehat{I}_i = Gr I_i, \ i = 1, \ldots, l$. The elements $\widehat{I}_i, \ i = 1, \ldots, l$ are called fundamental invariants.

Following Kostant we shall realize the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ as a subalgebra in $U(\mathfrak{b}_-)$. Let

$$\chi : \mathfrak{n}_+ \to \mathbb{C}$$

be a character of $\mathfrak{n}_+$. Since $\mathfrak{n}_+ = \sum_{i=1}^{l} \mathbb{C}X_{\alpha_i} \oplus [\mathfrak{n}_+, \mathfrak{n}_+]$ it is clear that χ is completely determined by the constants $c_i = \chi(X_{\alpha_i}), \ i = 1, \ldots, l$ and c_i are arbitrary. In [10] χ is called non–singular if $c_i \neq 0$ for all i.

Let $f = \sum_{i=1}^{l} X_{-\alpha_i} \in \mathfrak{n}_-$ be a regular nilpotent element. From the properties of the invariant bilinear form (see Section 1.1) it follows that $(f, [\mathfrak{n}_+, \mathfrak{n}_+]) = 0, \ (f, X_{\alpha_i}) = (X_{-\alpha_i}, X_{\alpha_i})$, and hence the map $x \mapsto (f, x), \ x \in \mathfrak{n}_+$ is a non–singular character of $\mathfrak{n}_+$.

Recall that in our choice of root vectors no normalization was made. But now given a non–singular character $\chi : \mathfrak{n}_+ \to \mathbb{C}$ we will say that f corresponds to χ in case

$$\chi(X_{\alpha_i}) = (X_{-\alpha_i}, X_{\alpha_i}).$$

Conversely if χ is non–singular there is a unique choice of f so that f corresponds to χ. In this case $\chi(x) = (f, x)$ for every $x \in \mathfrak{n}_+$.

Naturally, the character χ extends to a character of the universal enveloping algebra $U(\mathfrak{n}_+)$. Let $U_\chi(\mathfrak{n}_+)$ be the kernel of this extension so that one

has a direct sum

$$U(\mathfrak{n}_+) = \mathbb{C} \oplus U_\chi(\mathfrak{n}_+).$$

Since $\mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{n}_+$ we have a linear isomorphism $U(\mathfrak{g}) = U(\mathfrak{b}_-) \otimes U(\mathfrak{n}_+)$ and hence the direct sum

$$U(\mathfrak{g}) = U(\mathfrak{b}_-) \oplus I_\chi, \tag{1.1}$$

where $I_\chi = U(\mathfrak{g})U_\chi(\mathfrak{n}_+)$ is the left–sided ideal generated by $U_\chi(\mathfrak{n}_+)$.

For any $u \in U(\mathfrak{g})$ let $u^\chi \in U(\mathfrak{b}_-)$ be its component in $U(\mathfrak{b}_-)$ relative to the decomposition (1.1). Denote by ρ_χ the linear map

$$\rho_\chi : U(\mathfrak{g}) \to U(\mathfrak{b}_-)$$

given by $\rho_\chi(u) = u^\chi$. Let $W(\mathfrak{b}_-) = \rho_\chi(Z(\mathfrak{g}))$.

Theorem A ([10], Theorem 2.4.2) *The map*

$$\rho_\chi : Z(\mathfrak{g}) \to W(\mathfrak{b}_-) \tag{1.2}$$

is an isomorphism of algebras. In particular

$$W(\mathfrak{b}_-) = \mathbb{C}[I_1^\chi, \dots, I_l^\chi], \quad I_i^\chi = \rho_\chi(I_i), \quad i = 1, \dots, l$$

is a polynomial algebra in l generators.

Proof. First, we show that the map (1.2) is an algebra homomorphism. If $u, v \in Z(\mathfrak{g})$ then $u^\chi v^\chi \in U(\mathfrak{b}_-)$ and

$$uv - u^\chi v^\chi = (u - u^\chi)v + u^\chi(v - v^\chi).$$

Since $(u - u^\chi)v = v(u - u^\chi)$ the r.h.s. of the last equality is an element of I_χ. This proves $u^\chi v^\chi = (uv)^\chi$.

By definition the map (1.2) is surjective. We have to prove that it is injective. Let $U(\mathfrak{g})^\mathfrak{h}$ be the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$. Clearly $Z(\mathfrak{g}) \subseteq U(\mathfrak{g})^\mathfrak{h}$. From the Poincaré–Birkhoff–Witt theorem it follows that every element $z \in U(\mathfrak{g})^\mathfrak{h}$ may be uniquely written as

$$z = \sum_{p,q \in \mathbb{N}^N, <p>=<q>} X_{-\beta_1}^{p_1} \dots X_{-\beta_N}^{p_N} \varphi_{p,q} X_{\beta_1}^{q_1} \dots X_{\beta_N}^{q_N},$$

where $< p >= \sum_{i=1}^r p_i\beta_i \in \mathfrak{h}^*$ and $\varphi_{p,q} \in U(\mathfrak{h})$.

Now recall that $\chi(X_{\beta_i}) = 0$ if β_i is not a simple root, and we easily obtain

$$\rho_\chi(z) = \sum_{p,q\in\mathbb{N}^l, <p>=<q>\neq 0} X_{-\alpha_{k_1}}^{p_{j_1}} \dots X_{-\alpha_{k_l}}^{p_{j_l}} \varphi_{p,q} \prod_{i=1}^l c_{k_i}^{q_{j_i}} + \varphi_{0,0}.$$

Let $z \in Z(\mathfrak{g})$. One knows that the map

$$Z(\mathfrak{g}) \to U(\mathfrak{h}), \quad z \mapsto \varphi_{0,0},$$

called the Harich-Chandra homomorphism, is injective (see (c), p. 232 in [3]). It follows that the map (1.2) is also injective.

Remark 1.1 *The first part of the proof of Theorem A only used the fact that $v \in Z(\mathfrak{g})$. Therefore*

$$\rho_\chi(uv) = \rho_\chi(u)\rho_\chi(v)$$

for any $u \in U(\mathfrak{g}), \quad v \in Z(\mathfrak{g})$.

Definition A *The algebra $W(\mathfrak{b}_-)$ is called the Whittaker model of $Z(\mathfrak{g})$.*

Next we equip $U(\mathfrak{b}_-)$ with a structure of a left $U(\mathfrak{n}_+)$ module in such a way that $W(\mathfrak{b}_-)$ is realized as the space of invariants with respect to this action.

Let Y_χ be the left $U(\mathfrak{g})$ module defined by

$$Y_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}_+)} \mathbb{C}_\chi,$$

where $\mathbb{C}_\chi$ denotes the 1–dimensional $U(\mathfrak{n}_+)$–module defined by χ. Obviously Y_χ is just the quotient module $U(\mathfrak{g})/I_\chi$. From (1.1) it follows that the map

$$U(\mathfrak{b}_-) \to Y_\chi; \quad v \mapsto v \otimes 1 \tag{1.3}$$

is a linear isomorphism.

It is convenient to carry the module structure of Y_χ to $U(\mathfrak{b}_-)$. For $u \in U(\mathfrak{g}), \quad v \in U(\mathfrak{b}_-)$ the induced action $u \circ v$ has the form

$$u \circ v = (uv)^\chi. \tag{1.4}$$

The restriction of this action to $U(\mathfrak{n}_+)$ may be changed by tensoring with 1–dimensional $U(\mathfrak{n}_+)$–module defined by $-\chi$. That is $U(\mathfrak{b}_-)$ becomes an $U(\mathfrak{n}_+)$ module where if $x \in U(\mathfrak{n}_+)$, $v \in U(\mathfrak{b}_-)$ one puts

$$x \cdot v = x \circ v - \chi(x)v. \tag{1.5}$$

Lemma A ([10], Lemma 2.6.1.) *Let $v \in U(\mathfrak{b}_-)$ and $x \in U(\mathfrak{n}_+)$. Then*

$$x \cdot v = [x, v]^{\chi}.$$

Proof. By definition $x \cdot v = (xv)^{\chi} - \chi(x)v$. Then we have $xv = [x, v] + vx$ and hence $x \cdot v = ([x, v])^{\chi} + (vx)^{\chi} - \chi(x)v$. But clearly $(vx)^{\chi} = v\chi(x)$. Thus $x \cdot v = ([x, v])^{\chi}$.

The action (1.5) may be lifted to an action of the unipotent group N_+. Consider the space $U(\mathfrak{b}_-)^{N_+}$ of N_+ invariants in $U(\mathfrak{b}_-)$ with respect to this action. Clearly, $W(\mathfrak{b}_-) \subseteq U(\mathfrak{b}_-)^{N_+}$.

Theorem B ([10], Theorems 2.4.1, 2.6) *Suppose that the character χ is non–singular. Then the space of N_+ invariants in $U(\mathfrak{b}_-)$ with respect to the action (1.5) is isomorphic to $W(\mathfrak{b}_-)$, i.e.*

$$U(\mathfrak{b}_-)^{N_+} \cong W(\mathfrak{b}_-). \tag{1.6}$$

1.3 Whittaker modules

In this section we recall basic facts on Whittaker modules (see [10]).

Let V be a $U(\mathfrak{g})$ module. The action is denoted by uv for $u \in U(\mathfrak{g})$, $v \in V$. Let $\chi : \mathfrak{n}_+ \to \mathbb{C}$ be a non–singular character of $\mathfrak{n}_+$ (see Section 1.2). A vector $w \in V$ is called a Whittaker vector (with respect to χ) if

$$xw = \chi(x)w$$

for all $x \in U(\mathfrak{n}_+)$. A Whittaker vector w is called a cyclic Whittaker vector (for V) if $U(\mathfrak{g})w = V$. A $U(\mathfrak{g})$ module V is called a Whittaker module if it contains a cyclic Whittaker vector.

If V is any $U(\mathfrak{g})$ module we let $U_V(\mathfrak{g})$ be the annihilator of V. Then $U_V(\mathfrak{g})$ defines a central ideal $Z_V(\mathfrak{g})$ by putting

$$Z_V(\mathfrak{g}) = Z(\mathfrak{g}) \cap U_V(\mathfrak{g}). \tag{1.1}$$

Now assume that V is a Whittaker module for $U(\mathfrak{g})$ and $w \in V$ is a cyclic Whittaker vector. Let $U_w(\mathfrak{g}) \subseteq U(\mathfrak{g})$ be the annihilator of w. Thus $U_V(\mathfrak{g}) \subseteq U_w(\mathfrak{g})$, where $U_w(\mathfrak{g})$ is a left ideal and $U_V(\mathfrak{g})$ is a two–sided ideal in $U(\mathfrak{g})$. One has $V = U(\mathfrak{g})/U_w(\mathfrak{g})$ as $U(\mathfrak{g})$ modules so that V is determined up to equivalence by $U_w(\mathfrak{g})$. Clearly $I_\chi = U(\mathfrak{g})U_\chi(\mathfrak{n}_+) \subseteq U_w(\mathfrak{g})$ and $U(\mathfrak{g})Z_V(\mathfrak{g}) \subseteq U_w(\mathfrak{g})$.

The following theorem says that up to equivalence V is determined by the central ideal $Z_V(\mathfrak{g})$.

Theorem F ([10], Theorem 3.1) *Let V be any $U(\mathfrak{g})$ module which admits a cyclic Whittaker vector w and let $U_w(\mathfrak{g})$ be the annihilator of w. Then*

$$U_w(\mathfrak{g}) = U(\mathfrak{g})Z_V(\mathfrak{g}) + I_\chi. \tag{1.2}$$

The proof of Theorem F is based on the following lemma. We use the notation of Section 1.2. If $X \subseteq U(\mathfrak{g})$ let $X^\chi = \rho_\chi(X)$. Note that $U_w(\mathfrak{g})$ is stable under the map $u \mapsto \rho_\chi(u)$. We recall also that by Theorem A ρ_χ induces an algebra isomorphism $Z(\mathfrak{g}) \to W(\mathfrak{b}_-)$, where $W(\mathfrak{b}_-) = Z(\mathfrak{g})^\chi$. Thus if Z_* is any ideal in $Z(\mathfrak{g})$ then $W_*(\mathfrak{b}_-) = Z_*^\chi$ is an isomorphic ideal in $W(\mathfrak{b}_-)$. But $(U(\mathfrak{g})Z_*)^\chi = U(\mathfrak{b}_-)W_*(\mathfrak{b}_-)$ by Remark 1.1. Thus by (1.1) one has the direct sum

$$U(\mathfrak{g})Z_* + I_\chi = U(\mathfrak{b}_-)W_*(\mathfrak{b}_-) \oplus I_\chi. \tag{1.3}$$

Lemma B ([10], Lemma 3.1) *Let $X = \{v \in U(\mathfrak{b}_-) | (x \cdot v)w = 0$ for all $x \in \mathfrak{n}_+\}$, where $x \cdot v$ is given by (1.5). Then*

$$X = U(\mathfrak{b}_-)W_V(\mathfrak{b}_-) + W(\mathfrak{b}_-), \tag{1.4}$$

where $W_V(\mathfrak{b}_-) = Z_V(\mathfrak{g})^\chi$. Furthermore if we denote $U_w(\mathfrak{b}_-) = U_w(\mathfrak{g}) \cap U(\mathfrak{b}_-)$ then

$$U_w(\mathfrak{b}_-) = U(\mathfrak{b}_-)W_V(\mathfrak{b}_-). \tag{1.5}$$

Proof of Theorem F. Let $u \in U_w(\mathfrak{g})$. We wish to show that $u \in U(\mathfrak{g})Z_V(\mathfrak{g}) + I_\chi$. But by (1.3) it suffices to show that $u^\chi \in U(\mathfrak{b}_-)W_V(\mathfrak{b}_-)$. Since $u^\chi \in U_w(\mathfrak{b}_-)$ the result follows from (1.5).

Now one can determine, up to equivalence, the set of all Whittaker modules. They are naturally parametrized by the set of all ideals in the center $Z(\mathfrak{g})$.

Theorem G ([10], Theorem 3.2) *Let V be any Whittaker module for $U(\mathfrak{g})$, the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$. Let $U_V(\mathfrak{g})$ be the annihilator of V and let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Then the correspondence*

$$V \mapsto Z_V(\mathfrak{g}), \tag{1.6}$$

where $Z_V(\mathfrak{g}) = U_V(\mathfrak{g}) \cap Z(\mathfrak{g})$, sets up a bijection between the set of all equivalence classes of Whittaker modules and the set of all ideals in $Z(\mathfrak{g})$.

Proof. Let $V_i, \;\; i = 1, 2$ be two Whittaker modules. If $Z_{V_1}(\mathfrak{g}) = Z_{V_2}(\mathfrak{g})$ then clearly V_1 is equivalent to V_2 by (1.2). Thus the map (1.6) is injective on equivalence classes.

Conversely, let Z_* be any ideal in $Z(\mathfrak{g})$ and let $L = U(\mathfrak{g})Z_* + I_\chi$. Then $V = U(\mathfrak{g})/L$ is a Whittaker module, where we can take $U_w(\mathfrak{g}) = L$. But then $L = U(\mathfrak{g})Z_V(\mathfrak{g}) + I_\chi$ by Theorem F. By (1.2) this implies $Z_V(\mathfrak{g})^\chi = Z_*^\chi$. However by Theorem A ρ_χ is injective on $Z(\mathfrak{g})$. Therefore $Z_V(\mathfrak{g}) = Z_*$. Hence the map (1.6) is surjective.

Now consider the subalgebra $Z(\mathfrak{g})U(\mathfrak{n}_+)$ in $U(\mathfrak{g})$. By Theorem 2.1 in [11] $Z(\mathfrak{g})U(\mathfrak{n}_+) \cong Z(\mathfrak{g}) \otimes U(\mathfrak{n}_+)$. Now let Z_* be any ideal in $Z(\mathfrak{g})$ and regard $Z(\mathfrak{g})/Z_*$ as a $Z(\mathfrak{g})$ module. Equip $Z(\mathfrak{g})/Z_*$ with a structure of $Z(\mathfrak{g}) \otimes U(\mathfrak{n}_+)$ module by $u \otimes vy = \chi(v)uy$, where $u \in Z(\mathfrak{g}), \;\; v \in U(\mathfrak{n}_+), \;\; y \in Z(\mathfrak{g})/Z_*$. We denote this module by $(Z(\mathfrak{g})/Z_*)_\chi$

The following result is another way of expressing Theorem G.

Theorem H ([10], Theorem 3.3) *Let V be any $U(\mathfrak{g})$ module. Then V is a Whittaker module if and only if one has an isomorphism*

$$V \cong U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}) \otimes U(\mathfrak{n}_+)} (Z(\mathfrak{g})/Z_*)_\chi \tag{1.7}$$

of $U(\mathfrak{g})$ modules. Furthermore in such a case the ideal Z_* is unique and is given by $Z_* = Z_V(\mathfrak{g})$, where $Z_V(\mathfrak{g})$ is defined by (1.1).

*Proof.*If 1_* is the image of 1 in $Z(\mathfrak{g})/Z_*$ then the annihilator in $Z(\mathfrak{g}) \otimes U(\mathfrak{n}_+)$ of 1_* is $U(\mathfrak{n}_+)Z_* + Z(\mathfrak{g})U_\chi(\mathfrak{n}_+)$. Thus the annihilator in $U(\mathfrak{g})$ of $1 \otimes 1_* = w$ in the right side of (1.7) is $U(\mathfrak{g})Z_* + I_\chi$. The result then follows from Theorem G since w is clearly a cyclic generator of this module.

Now one can determine all the Whittaker vectors in a Whittaker module and duscuss the question of irreducibility for Whittaker modules.

Theorem K ([10], Theorem 3.4) *Let V be any $U(\mathfrak{g})$ module with a cyclic Whittaker vector $w \in V$. Then any $v \in V$ is a Whittaker vector if and only if v is of the form $v = uw$, where $u \in Z(\mathfrak{g})$. Thus the space of all Whittaker vectors in V is a cyclic $Z(\mathfrak{g})$ module which is isomorphic to $Z(\mathfrak{g})/Z_V(\mathfrak{g})$.*

Proof. Obviously if $v = uw$ for $u \in Z(\mathfrak{g})$ then v is a Whittaker vector. Conversely let $v \in V$ be a Whittaker vector. Write $v = uw$ for $u \in U(\mathfrak{g})$. Then clearly $v = u^\chi w$ so that we can assume $u \in U(\mathfrak{b}_-)$. But now if $x \in \mathfrak{n}_+$ then $xuw = \chi(x)uw$. But also $usw = \chi(x)uw$. Thus $[x, u]w = 0$ and hence $[x, u]^\chi w = 0$. But $x \cdot u = [x, u]^\chi$ by Lemma A. Thus in the notation of Lemma B one has $u \in X$. But then by Lemma B one can write $u = u_1 + u_2$, where $u_1 \in U_w(\mathfrak{b}_-)$ and $u_2 \in W(\mathfrak{b}_-)$. But then $u_1 w = 0$. Thus $v = u_2 w$. But now $u_2 = u_3^\chi$, where $u_3 \in Z(\mathfrak{g})$ by Theorem A. But then $v = u_3 w$ which proves the theorem.

If V is any $U(\mathfrak{g})$ module then $\mathrm{End}_U\ V$ denotes the algebra of operators on V which commute with the action of $U(\mathfrak{g})$. If $\pi_V : U(\mathfrak{g}) \to \mathrm{End}\ V$ is the representation defining the $U(\mathfrak{g})$ module structure on V then clearly $\pi_V(Z(\mathfrak{g})) \subseteq \mathrm{End}_U\ V$. Furthermore it is also clear that $\pi_V(Z(\mathfrak{g})) \cong Z(\mathfrak{g})/Z_V(\mathfrak{g})$.

Theorem L ([10], Theorem 3.5) *Assume that V is a Whittaker module. Then $\mathrm{End}_U\ V = \pi_V(Z(\mathfrak{g}))$. In particular one has an isomorphism*

$$\mathrm{End}_U\ V \cong Z(\mathfrak{g})/Z_V(\mathfrak{g}).$$

Note that $\mathrm{End}_U\ V$ is commutative.

Proof. Let $w \in V$ be a cyclic Whittaker vector. If $\alpha \in \mathrm{End}_U\ V$ then αw is a Whittaker vector. But then by Theorem K there exists $u \in Z(\mathfrak{g})$ such that $\alpha w = uw$. For any $v \in U(\mathfrak{g})$ one has $\alpha vw = v\alpha w = vuw = uvw$. Thus $\alpha = \pi_V(u)$.

Now one can describe all irreducible Whittaker modules. A homomorphism

$$\xi : Z(\mathfrak{g}) \to \mathbb{C}$$

is called a central character. Given a central character ξ let $Z_\xi(\mathfrak{g}) = \mathrm{Ker}\ \xi$ so that $Z_\xi(\mathfrak{g})$ is a typical central ideal in $Z(\mathfrak{g})$.

If V is any $U(\mathfrak{g})$ module one says that V admits an infinitesimal character, and ξ is its infinitesimal character, if ξ is a central character such that $uv = \xi(u)v$ for all $u \in Z(\mathfrak{g})$, $v \in V$. Recall that by Dixmier's theorem any irreducible $U(\mathfrak{g})$ module admits an infinitesimal character.

Given a central character ξ let $\mathbb{C}_{\xi,\chi}$ be the 1-dimensional $Z(\mathfrak{g}) \otimes U(\mathfrak{n}_+)$ module defined so that if $u \in Z(\mathfrak{g})$, $v \in U(\mathfrak{n}_+)$, $y \in \mathbb{C}_{\xi,\chi}$ then $u \otimes vy = \xi(u)\chi(v)y$. Also let

$$Y_{\xi,\chi} = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})\otimes U(\mathfrak{n}_+)} \mathbb{C}_{\xi,\chi}.$$

It is clear that $Y_{\xi,\chi}$ admits an infinitesimal character and ξ is that character.

Theorem M ([10], Theorem 3.6.1) *Let V be any Whittaker module for $U(\mathfrak{g})$, the universal enveloping of a simple Lie algebra $\mathfrak{g}$. Then the following conditions are equivalent:*

(1) V is an irreducible $U(\mathfrak{g})$ module.

(2) V admits an infinitesimal character.

(3) The corresponding ideal given by Theorem G is a maximal ideal.

(4) The space of Whittaker vectors in V is 1–dimensional.

(5) All non–zero Whittaker vectors in V are cyclic vectors.

(6) The centralizer $\mathrm{End}_U\, V$ reduces to constants $\mathbb{C}$.

(7) V is isomorphic to $Y_{\xi,\chi}$ for some central character ξ.

Proof. The equivalence of (2), (3), (4) and (6) follows from Theorems K and L. This is also equivalent to (5) since (5) implies that $Z(\mathfrak{g})/Z_V(\mathfrak{g})$ is a field by Theorem K. One gets the equivalence with (7) by Theorem H. It remains to relate (2)–(7) with (1). But (1) implies (2) by Dixmier's theorem. The proof of the equivalence (2)–(7) with (1) may be found in [10].

2 Quantum deformation of the Whittaker model

Let $\mathfrak{g}$ be a complex simple Lie algebra, $U_h(\mathfrak{g})$ the standard quantum group associated with $\mathfrak{g}$. In this section we construct a generalization of the Whittaker model $W(\mathfrak{b}_-)$ for $U_h(\mathfrak{g})$.

Let $U_h(\mathfrak{n}_+)$ be the subalgebra of $U_h(\mathfrak{g})$ corresponding to the nilpotent Lie subalgebra $\mathfrak{n}_+$. $U_h(\mathfrak{n}_+)$ is generated by simple positive root generators of $U_h(\mathfrak{g})$ subject to the quantum Serre relations. It is easy to show that $U_h(\mathfrak{n}_+)$ has no non–singular characters (taking nonvanishing values on all simple

root generators). Our first main result is a family of new realizations of the quantum group $U_h(\mathfrak{g})$, one for each Coxeter element in the corresponding Weyl group (see also [13]). The counterparts of $U(\mathfrak{n}_+)$, which naturally arise in these new realizations of $U_h(\mathfrak{g})$, do have non–singular characters.

Using these new realizations we can immediately formulate a quantum group version of Definition A. We also prove counterparts of Theorems A and B for $U_h(\mathfrak{g})$.

Finally we define quantum group generalizations of the Toda Hamiltonians. In the spirit of quantum harmonic analysis these new Hamiltonians are difference operators. An alternative definition of these Hamiltonians has been recently given in [6].

2.1 Quantum groups

In this section we recall some basic facts about quantum groups. We follow the notation of [2].

Let h be an indeterminate, $\mathbb{C}[[h]]$ the ring of formal power series in h. We shall consider $\mathbb{C}[[h]]$–modules equipped with the so–called h–adic topology. For every such module V this topology is characterized by requiring that $\{h^nV \mid n \geq 0\}$ is a base of the neighbourhoods of 0 in V, and that translations in V are continuous. It is easy to see that, for modules equipped with this topology, every $\mathbb{C}[[h]]$–module map is automatically continuous.

A topological Hopf algebra over $\mathbb{C}[[h]]$ is a complete $\mathbb{C}[[h]]$–module A equipped with a structure of $\mathbb{C}[[h]]$–Hopf algebra (see [2], Definition 4.3.1), the algebraic tensor products entering the axioms of the Hopf algebra are replaced by their completions in the h–adic topology. We denote by $\mu, \imath, \Delta, \varepsilon, S$ the multiplication, the unit, the comultiplication, the counit and the antipode of A, respectively.

The standard quantum group $U_h(\mathfrak{g})$ associated to a complex finite–dimensional simple Lie algebra $\mathfrak{g}$ is the algebra over $\mathbb{C}[[h]]$ topologically generated by elements $H_i,\ X_i^+,\ X_i^-,\ i = 1, \ldots, l$, and with the following defining relations:

$$[H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm,$$

$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{2.1}$$

where $K_i = e^{d_i h H_i}, \quad e^h = q, \quad q_i = q^{d_i} = e^{d_i h}$,

and the quantum Serre relations:

$$\sum_{r=0}^{1-a_{ij}}(-1)^r \left[\begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} (X_i^{\pm})^{1-a_{ij}-r} X_j^{\pm} (X_i^{\pm})^r = 0,\ i \neq j,$$

where

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q![n-m]_q!},\ [n]_q! = [n]_q \dots [1]_q,\ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

$U_h(\mathfrak{g})$ is a topological Hopf algebra over $\mathbb{C}[[h]]$ with comultiplication defined by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_h(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+,$$

$$\Delta_h(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-,$$

antipode defined by

$$S_h(H_i) = -H_i,\ \ S_h(X_i^+) = -X_i^+ K_i^{-1},\ \ S_h(X_i^-) = -K_i X_i^-,$$

and counit defined by

$$\varepsilon_h(H_i) = \varepsilon_h(X_i^{\pm}) = 0.$$

We shall also use the weight–type generators defined by

$$Y_i = \sum_{j=1}^{l} d_i (a^{-1})_{ij} H_j,$$

and the elements $L_i = e^{hY_i}$. They commute with the root vectors $X_i^{\pm}$ as follows:

$$L_i X_j^{\pm} L_i^{-1} = q_i^{\pm \delta_{ij}} X_j^{\pm}. \tag{2.2}$$

The Hopf algebra $U_h(\mathfrak{g})$ is a quantization of the standard bialgebra structure on $\mathfrak{g}$, i.e. $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g})$, $\Delta_h = \Delta \pmod h$, where Δ is the standard comultiplication on $U(\mathfrak{g})$, and

$$\frac{\Delta_h - \Delta_h^{opp}}{h} \pmod h = \delta,$$

where $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the standard cocycle on $\mathfrak{g}$. Recall that

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) 2r_+, \quad r_+ \in \mathfrak{g} \otimes \mathfrak{g},$$

$$r_+ = \frac{1}{2} \sum_{i=1}^{l} Y_i \otimes X_i + \sum_{\beta \in \Delta_+} (X_\beta, X_{-\beta})^{-1} X_\beta \otimes X_{-\beta}. \tag{2.3}$$

Here $X_{\pm\beta} \in \mathfrak{g}_{\pm\beta}$ are root vectors of $\mathfrak{g}$. The element $r_+ \in \mathfrak{g} \otimes \mathfrak{g}$ is called a classical r–matrix.

The following proposition describes the algebraic structure of $U_h(\mathfrak{g})$.

Proposition 2.1 ([2], Proposition 6.5.5) *Let $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra, let $U_h(\mathfrak{h})$ be the subalgebra of $U_h(\mathfrak{g})$ topologically generated by the $H_i, i = 1, \ldots l$. Then, there is an isomorphism of algebras $\varphi : U_h(\mathfrak{g}) \to U(\mathfrak{g})[[h]]$ over $\mathbb{C}[[h]]$ such that $\varphi = id$ (mod h) and $\varphi|_{U_h(\mathfrak{h})} = id$.*

Proposition 2.2 ([2], Proposition 6.5.7) *If $\mathfrak{g}$ is a finite–dimensional complex simple Lie algebra, the center $Z_h(\mathfrak{g})$ of $U_h(\mathfrak{g})$ is canonically isomorphic to $Z(\mathfrak{g})[[h]]$, where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$.*

Corollary 2.3 ([2], Corollary 6.5.6) *If $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra, then the assignment $V \mapsto V[[h]]$ is a one–to–one correspondence between the finite–dimensional irreducible representations of $\mathfrak{g}$ and indecomposable representations of $U_h(\mathfrak{g})$ which are free and of finite rank as $\mathbb{C}[[h]]$–modules. Furthermore for every such V the action of the generators $H_i \in U_h(\mathfrak{g}), \quad i = 1, \ldots l$ on $V[[h]]$ coincides with the action of the root generators $H_i \in \mathfrak{h}, \quad i = 1, \ldots l$.*

The representations of $U_h(\mathfrak{g})$ defined in the previous corollary are called finite–dimensional representations of $U_h(\mathfrak{g})$. For every finite–dimensional representation $\pi_V : \mathfrak{g} \to \mathrm{End}V$ we denote the corresponding representation of $U_h(\mathfrak{g})$ in the space $V[[h]]$ by the same letter.

$U_h(\mathfrak{g})$ is a quasitriangular Hopf algebra, i.e. there exists an invertible element $\mathcal{R} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$, called a universal R–matrix, such that

$$\Delta_h^{opp}(a) = \mathcal{R} \Delta_h(a) \mathcal{R}^{-1} \text{ for all } a \in U_h(\mathfrak{g}), \tag{2.4}$$

where $\Delta^{opp} = \sigma\Delta$, σ is the permutation in $U_h(\mathfrak{g})^{\otimes 2}$, $\sigma(x \otimes y) = y \otimes x$, and

$$\begin{aligned}(\Delta_h \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23},\\ (id \otimes \Delta_h)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12},\end{aligned} \tag{2.5}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1,\ \mathcal{R}_{23} = 1 \otimes \mathcal{R},\ \mathcal{R}_{13} = (\sigma \otimes id)\mathcal{R}_{23}$.

From (2.4) and (2.5) it follows that $\mathcal{R}$ satisfies the quantum Yang–Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.6}$$

For every quasitriangular Hopf algebra we also have (see Proposition 4.2.7 in [2]):

$$(S \otimes id)\mathcal{R} = \mathcal{R}^{-1},$$

and

$$(S \otimes S)\mathcal{R} = \mathcal{R}. \tag{2.7}$$

We shall explicitly describe the element $\mathcal{R}$. First following [9] we recall the construction of root vectors of $U_h(\mathfrak{g})$. We shall use the so–called normal ordering in the root system $\Delta_+ = \{\beta_1, \ldots, \beta_N\}$ (see [14]).

Definition 2.1 *An ordering of the root system Δ_+ is called normal if all simple roots are written in an arbitrary order, and for any theree roots $\alpha,\ \beta,\ \gamma$ such that $\gamma = \alpha + \beta$ we have either $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$.*

To construct root vectors we shall apply the following inductive algorithm. Let $\alpha, \beta, \gamma \in \Delta_+$ be positive roots such that $\gamma = \alpha + \beta,\ \alpha < \beta$ and $[\alpha, \beta]$ is the minimal segment including γ, i.e. the segment has no other roots α', β' such that $\gamma = \alpha' + \beta'$. Suppose that $X^{\pm}_{\alpha},\ X^{\pm}_{\beta}$ have already been constructed. Then we define

$$\begin{aligned}X^+_{\gamma} = X^+_{\alpha}X^+_{\beta} - q^{(\alpha,\beta)}X^+_{\beta}X^+_{\alpha},\\ X^-_{\gamma} = X^-_{\beta}X^-_{\alpha} - q^{-(\alpha,\beta)}X^-_{\alpha}X^-_{\beta}.\end{aligned} \tag{2.8}$$

Proposition 2.4 *For $\beta = \sum_{i=1}^{l} m_i\alpha_i,\ m_i \in \mathbb{N}$ $X^{\pm}_{\beta}$ is a polynomial in the noncommutative variables $X^{\pm}_i$ homogeneous in each $X^{\pm}_i$ of degree m_i.*

The root vectors X_β satisfy the following relations:

$$[X_\alpha^+, X_\alpha^-] = a(\alpha)\frac{e^{h\alpha^\vee} - e^{-h\alpha^\vee}}{q - q^{-1}}.$$

where $a(\alpha) \in \mathbb{C}[[h]]$. They commute with elements of the subalgebra $U_h(\mathfrak{h})$ as follows:

$$[H_i, X_\beta^\pm] = \pm\beta(H_i)X_\beta^\pm,\ i = 1, \ldots, l. \tag{2.9}$$

Note that by construction

$$X_\beta^+ \pmod h = X_\beta \in \mathfrak{g}_\beta,$$

$$X_\beta^- \pmod h = X_{-\beta} \in \mathfrak{g}_{-\beta}$$

are root vectors of $\mathfrak{g}$. This implies that $a(\alpha) \pmod h = (X_\alpha, X_{-\alpha})$.

Let $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ be the $\mathbb{C}[[h]]$–subalgebras of $U_h(\mathfrak{g})$ topologically generated by the X_i^+ and by the X_i^-, respectively.

Now using the root vectors $X_\beta^\pm$ we can construct a topological basis of $U_h(\mathfrak{g})$. Define for $\mathbf{r} = (r_1, \ldots, r_N) \in \mathbb{N}^N$,

$$(X^+)^{\mathbf{r}} = (X_{\beta_1}^+)^{r_1} \ldots (X_{\beta_N}^+)^{r_N},$$

$$(X^-)^{\mathbf{r}} = (X_{\beta_1}^-)^{r_1} \ldots (X_{\beta_N}^-)^{r_N},$$

and for $\mathbf{s} = (s_1, \ldots s_l) \in \mathbb{N}^{\,l}$,

$$H^{\mathbf{s}} = H_1^{s_1} \ldots H_l^{s_l}.$$

Proposition 2.5 ([9], Proposition 3.3) *The elements $(X^+)^{\mathbf{r}}$, $(X^-)^{\mathbf{t}}$ and $H^{\mathbf{s}}$, for $\mathbf{r},\ \mathbf{t} \in \mathbb{N}^N$, $\mathbf{s} \in \mathbb{N}^l$, form topological bases of $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ and $U_h(\mathfrak{h})$, respectively, and the products $(X^+)^{\mathbf{r}}H^{\mathbf{s}}(X^-)^{\mathbf{t}}$ form a topological basis of $U_h(\mathfrak{g})$. In particular, multiplication defines an isomorphism of $\mathbb{C}[[h]]$ modules:*

$$U_h(\mathfrak{n}_-) \otimes U_h(\mathfrak{h}) \otimes U_h(\mathfrak{n}_+) \to U_h(\mathfrak{g}).$$

An explicit expression for $\mathcal{R}$ may be written by making use of the q-exponential

$$exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k)_q!},$$

where

$$(k)_q! = (1)_q \dots (k)_q, \quad (n)_q = \frac{q^n - 1}{q - 1}.$$

Now the element $\mathcal{R}$ may be written as (see Theorem 8.1 in [9]):

$$\mathcal{R} = exp\left[h \sum_{i=1}^{l} (Y_i \otimes H_i)\right] \prod_{\beta} exp_{q_\beta^{-1}}[(q - q^{-1})a(\beta)^{-1} X_\beta^+ \otimes X_\beta^-], \qquad (2.10)$$

where $q_\beta = q^{(\beta,\beta)}$; the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that the α–term appears to the left of the β–term if $\alpha < \beta$ with respect to the normal ordering of Δ_+.

Remark 2.2 *The r–matrix $r_+ = \frac{1}{2}h^{-1}(\mathcal{R} - 1 \otimes 1) \pmod h$, which is the classical limit of $\mathcal{R}$, coincides with the classical r–matrix (2.3).*

2.2 Non–singular characters and quantum groups

In this section following [13] we recall the construction of quantum counterparts of the principal nilpotent Lie subalgebras of complex simple Lie algebras and of their non–singular characters. Subalgebras of $U_h(\mathfrak{g})$ which resemble the subalgebra $U(\mathfrak{n}_+) \subset U(\mathfrak{g})$ and possess non–singular characters naturally appear in the Coxeter realizations of $U_h(\mathfrak{g})$ defined in [13] as follows.

Denote by S_l the symmetric group of l elements. To any element $\pi \in S_l$ we associate a Coxeter element s_π by the formula $s_\pi = s_{\pi(1)} \dots s_{\pi(l)}$. Let $U_h^{s_\pi}(\mathfrak{g})$ be the associative algebra over $\mathbb{C}[[h]]$ with generators $e_i, f_i, H_i,\ i = 1, \dots l$

subject to the relations:

$$
\begin{aligned}
&[H_i,H_j]=0,\ \ [H_i,e_j]=a_{ij}e_j,\ \ [H_i,f_j]=-a_{ij}f_j,\\
&e_if_j-q^{c^\pi_{ij}}f_je_i=\delta_{i,j}\tfrac{K_i-K_i^{-1}}{q_i-q_i^{-1}},\\
&K_i=e^{d_ihH_i},\\
&\textstyle\sum_{r=0}^{1-a_{ij}}(-1)^rq^{rc^\pi_{ij}}\left[\begin{array}{c}1-a_{ij}\\ r\end{array}\right]_{q_i}(e_i)^{1-a_{ij}-r}e_j(e_i)^r=0,\ i\neq j,\\
&\textstyle\sum_{r=0}^{1-a_{ij}}(-1)^rq^{rc^\pi_{ij}}\left[\begin{array}{c}1-a_{ij}\\ r\end{array}\right]_{q_i}(f_i)^{1-a_{ij}-r}f_j(f_i)^r=0,\ i\neq j,
\end{aligned}
\tag{2.1}
$$

where $c^\pi_{ij}=\left(\frac{1+s_\pi}{1-s_\pi}\alpha_i,\alpha_j\right)$ are matrix elements of the Cayley transform of s_π in the basis of simple roots.

Theorem 2.1 ([13], Theorem 4) *For every solution $n_{ij}\in\mathbb{C},\ i,j=1,\ldots,l$ of equations*

$$d_jn_{ij}-d_in_{ji}=c^\pi_{ij} \tag{2.2}$$

there exists an algebra isomorphism $\psi_{\{n\}}:U_h^{s_\pi}(\mathfrak{g})\rightarrow U_h(\mathfrak{g})$ defined by the formulas:

$$
\begin{aligned}
&\psi_{\{n\}}(e_i)=X_i^+\textstyle\prod_{p=1}^lL_p^{n_{ip}},\\
&\psi_{\{n\}}(f_i)=\textstyle\prod_{p=1}^lL_p^{-n_{ip}}X_i^-,\\
&\psi_{\{n\}}(H_i)=H_i.
\end{aligned}
$$

We call the algebra $U_h^{s_\pi}(\mathfrak{g})$ the Coxeter realization of the quantum group $U_h(\mathfrak{g})$ corresponding to the Coxeter element s_π.

Let $U_h^{s_\pi}(\mathfrak{n}_+)$ be the subalgebra in $U_h^{s_\pi}(\mathfrak{g})$ generated by $e_i, i=1,\ldots l$.

Proposition 2.2 ([13], Proposition 2) *The map $\chi_h^{s_\pi}:U_h^{s_\pi}(\mathfrak{n}_+)\rightarrow\mathbb{C}[[h]]$ defined on generators by $\chi_h^{s_\pi}(e_i)=c_i,\ c_i\in\mathbb{C}[[h]],\ c_i\neq 0$ is a character of the algebra $U_h^{s_\pi}(\mathfrak{n}_+)$.*

The proof of this proposition given in [13] is based on the following Lemma.

Lemma 2.3 ([13], Lemma 3) *The matrix elements of $\frac{1+s_\pi}{1-s_\pi}$ are of the form :*

$$c_{ij}^{\pi} = \left(\frac{1+s_\pi}{1-s_\pi}\alpha_i, \alpha_j\right) = \varepsilon_{ij}^{\pi} b_{ij}, \tag{2.3}$$

where

$$\varepsilon_{ij}^{\pi} = \begin{cases} -1 & \pi^{-1}(i) < \pi^{-1}(j) \\ 0 & i = j \\ 1 & \pi^{-1}(i) > \pi^{-1}(j) \end{cases}$$

Now we shall study the algebraic structure of $U_h^{s_\pi}(\mathfrak{g})$. Denote by $U_h^{s_\pi}(\mathfrak{n}_-)$ the subalgebra in $U_h^{s_\pi}(\mathfrak{g})$ generated by $f_i, i = 1, \ldots l$. From defining relations (2.1) it follows that the map $\overline{\chi}_h^{s_\pi} : U_h^{s_\pi}(\mathfrak{n}_-) \to \mathbb{C}[[h]]$ defined on generators by $\overline{\chi}_h^{s_\pi}(f_i) = c_i, c_i \in \mathbb{C}[[h]], c_i \neq 0$ is a character of the algebra $U_h^{s_\pi}(\mathfrak{n}_-)$.

Let $U_h^{s_\pi}(\mathfrak{h})$ be the subalgebra in $U_h^{s_\pi}(\mathfrak{g})$ generated by $H_i,\ i = 1, \ldots, l$. Define $U_h^{s_\pi}(\mathfrak{b}_\pm) = U_h^{s_\pi}(\mathfrak{n}_\pm)U_h^{s_\pi}(\mathfrak{h})$.

We shall construct a Poincaré–Birkhoff-Witt basis for $U_h^{s_\pi}(\mathfrak{g})$. It is convenient to introduce an operator $K \in \mathrm{End}\ \mathfrak{h}$ such that

$$KH_i = \sum_{j=1}^{l} \frac{n_{ij}}{d_i} Y_j. \tag{2.4}$$

In particular, we have

$$\frac{n_{ji}}{d_j} = (KH_j, H_i).$$

Equation (2.2) is equivalent to the following equation for the operator K:

$$K - K^* = \frac{1+s_\pi}{1-s_\pi}.$$

Proposition 2.4 *(i)For any solution of equation (2.2) and any normal ordering of the root system Δ_+ the elements $e_\beta = \psi_{\{n\}}^{-1}(X_\beta^+ e^{hK\beta^\vee})$ and $f_\beta = \psi_{\{n\}}^{-1}(e^{-hK\beta^\vee} X_\beta^-)$, $\beta \in \Delta_+$ lie in the subalgebras $U_h^{s_\pi}(\mathfrak{n}_+)$ and $U_h^{s_\pi}(\mathfrak{n}_-)$, respectively.*

(ii)Moreover, the elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1} \ldots e_{\beta_N}^{r_N}$, $f^{\mathbf{t}} = e_{\beta_1}^{t_1} \ldots e_{\beta_N}^{t_N}$ *and* $H^{\mathbf{s}} = H_1^{s_1} \ldots H_l^{s_l}$ *for* $\mathbf{r}, \mathbf{t}, \mathbf{s} \in \mathbb{N}^N$, *form topological bases of* $U_h^{s_\pi}(\mathfrak{n}_+)$, $U_h^{s_\pi}(\mathfrak{n}_-)$ *and* $U_h^{s_\pi}(\mathfrak{h})$, *and the products* $f^{\mathbf{t}} H^{\mathbf{s}} e^{\mathbf{r}}$ *form a topological basis of* $U_h^{s_\pi}(\mathfrak{g})$. *In particular, multiplication defines an isomorphism of* $\mathbb{C}[[h]]$ *modules*

$$U_h^{s_\pi}(\mathfrak{n}_-) \otimes U_h^{s_\pi}(\mathfrak{h}) \otimes U_h^{s_\pi}(\mathfrak{n}_+) \to U_h^{s_\pi}(\mathfrak{g}).$$

Proof. Let $\beta = \sum_{i=1}^l m_i \alpha_i \in \Delta_+$ be a positive root, $X_\beta^+ \in U_h(\mathfrak{g})$ the corresponding root vector. Then $\beta^\vee = \sum_{i=1}^l m_i d_i H_i$, and so $K\beta^\vee = \sum_{i,j=1}^l m_i n_{ij} Y_j$. Now the proof of the first statement follows immediately from Proposition 2.4, commutation relations (2.2) and the definition of the isomorphism $\psi_{\{n\}}$. The second assertion is a consequence of Proposition 2.5.

Now we would like to choose a normal ordering of the root system Δ_+ in such a way that $\chi_h^{s_\pi}(e_\beta) = 0$ and $\overline{\chi}_h^{s_\pi}(f_\beta) = 0$ if β is not a simple root.

Proposition 2.5 *Choose a normal ordering of the root system* Δ_+ *such that the simple roots are written in the following order:* $\alpha_{\pi(1)}, \ldots, \alpha_{\pi(l)}$. *Then* $\chi_h^{s_\pi}(e_\beta) = 0$ *and* $\overline{\chi}_h^{s_\pi}(f_\beta) = 0$ *if* β *is not a simple root.*

Proof. We shall consider the case of positive root generators. The proof for negative root generators is similar to that for the positive ones.

The root vectors X_β^+ are defined in terms of iterated q-commutators (see (2.8)). Therefore it suffices to verify that for $i < j$

$$\chi_h^{s_\pi}(e_{\alpha_{\pi(i)}+\alpha_{\pi(j)}}) =$$

$$\chi_h^{s_\pi}(\psi_{\{n\}}^{-1}((X_{\pi(i)}^+ X_{\pi(j)}^+ - q^{(\alpha_{\pi(i)},\alpha_{\pi(j)})} X_{\pi(j)}^+ X_{\pi(i)}^+) e^{hK(d_{\pi(i)}H_{\pi(i)} + d_{\pi(j)}H_{\pi(j)})})) = 0.$$

From (2.4) and commutation relations (2.2) we obtain that

$$\begin{gathered}\psi_{\{n\}}^{-1}((X_{\pi(i)}^+ X_{\pi(j)}^+ - q^{(\alpha_{\pi(i)},\alpha_{\pi(j)})} X_{\pi(j)}^+ X_{\pi(i)}^+) e^{hK(d_{\pi(i)}H_{\pi(i)} + d_{\pi(j)}H_{\pi(j)})}) = \\ q^{-d_{\pi(j)} n_{\pi(i)\pi(j)}} \left(e_{\pi(i)} e_{\pi(j)} - q^{b_{\pi(i)\pi(j)} + d_{\pi(j)} n_{\pi(i)\pi(j)} - d_{\pi(i)} n_{\pi(j)\pi(i)}} e_{\pi(j)} e_{\pi(i)}\right)\end{gathered} \tag{2.5}$$

Now using equation (2.2) and Lemma 2.3 the combination $b_{\pi(i)\pi(j)} + d_{\pi(j)} n_{\pi(i)\pi(j)} - d_{\pi(i)} n_{\pi(j)\pi(i)}$ may be represented as $b_{\pi(i)\pi(j)} + \varepsilon_{\pi(i)\pi(j)}^\pi b_{\pi(i)\pi(j)}$. But $\varepsilon_{\pi(i)\pi(j)}^\pi = -1$ for $i < j$ and therefore the r.h.s. of (2.5) takes the form

$$q^{-d_{\pi(j)} n_{\pi(i)\pi(j)}} [e_{\pi(i)}, e_{\pi(j)}].$$

Clearly,

$$\chi_h^{s_\pi}(e_{\alpha_{\pi(i)}+\alpha_{\pi(j)}}) = q^{-d_{\pi(j)} n_{\pi(i)\pi(j)}} \chi_h^{s_\pi}([e_{\pi(i)}, e_{\pi(j)}]) = 0.$$

2.3 Quantum deformation of the Whittaker model

In this section we define a quantum deformation of the Whittaker model $W(\mathfrak{b}_-)$. Our construction is similar the one described in Section 1.2, the quantum group $U_h^{s_\pi}(\mathfrak{g})$, the subalgebra $U_h^{s_\pi}(\mathfrak{n}_+)$ and characters $\chi_h^{s_\pi} : U_h^{s_\pi}(\mathfrak{n}_+) \to \mathbb{C}[[h]]$ serve as natural counterparts of the universal enveloping algebra $U(\mathfrak{g})$, of the subalgebra $U(\mathfrak{n}_+)$ and of non–singular characters $\chi : U(\mathfrak{n}_+) \to \mathbb{C}$.

Let $U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}}$ be the kernel of the character $\chi_h^{s_\pi} : U_h^{s_\pi}(\mathfrak{n}_+) \to \mathbb{C}[[h]]$ so that one has a direct sum

$$U_h^{s_\pi}(\mathfrak{n}_+) = \mathbb{C}[[h]] \oplus U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}}.$$

From Proposition 2.4 we have a linear isomorphism $U_h^{s_\pi}(\mathfrak{g}) = U_h^{s_\pi}(\mathfrak{b}_-) \otimes U_h^{s_\pi}(\mathfrak{n}_+)$ and hence the direct sum

$$U_h^{s_\pi}(\mathfrak{g}) = U_h^{s_\pi}(\mathfrak{b}_-) \oplus I_{\chi_h^{s_\pi}}, \tag{2.1}$$

where $I_{\chi_h^{s_\pi}} = U_h^{s_\pi}(\mathfrak{g})U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}}$ is the left–sided ideal generated by $U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}}$.

For any $u \in U_h^{s_\pi}(\mathfrak{g})$ let $u^{\chi_h^{s_\pi}} \in U_h^{s_\pi}(\mathfrak{b}_-)$ be its component in $U_h^{s_\pi}(\mathfrak{b}_-)$ relative to the decomposition (2.1). Denote by $\rho_{\chi_h^{s_\pi}}$ the linear map

$$\rho_{\chi_h^{s_\pi}} : U_h^{s_\pi}(\mathfrak{g}) \to U_h^{s_\pi}(\mathfrak{b}_-)$$

given by $\rho_{\chi_h^{s_\pi}}(u) = u^{\chi_h^{s_\pi}}$.

Denote by $Z_h^{s_\pi}(\mathfrak{g})$ the center of $U_h^{s_\pi}(\mathfrak{g})$. From Proposition 2.2 and Theorem 2.1 we obtain that $Z_h^{s_\pi}(\mathfrak{g}) \cong Z(\mathfrak{g})[[h]]$. In particular, $Z_h^{s_\pi}(\mathfrak{g})$ is freely generated as a commutative topological algebra over $\mathbb{C}[[h]]$ by l elements $I_1, \ldots, I_l$.

Let $W_h(\mathfrak{b}_-) = \rho_{\chi_h^{s_\pi}}(Z_h^{s_\pi}(\mathfrak{g}))$.

Theorem $\mathbf{A_h}$ *The map*

$$\rho_{\chi_h^{s_\pi}} : Z_h^{s_\pi}(\mathfrak{g}) \to W_h(\mathfrak{b}_-) \tag{2.2}$$

is an isomorphism of algebras. In particular, $W_h(\mathfrak{b}_-)$ is freely generated as a commutative topological algebra over $\mathbb{C}[[h]]$ by l elements $I_i^{\chi_h^{s_\pi}} = \rho_{\chi_h^{s_\pi}}(I_i),\ \ i = 1, \ldots, l$.

Proof is similar to that of Theorem A in the classical case.

Remark 2.3 *Similarly to Remark 1.1 we have*

$$\rho_\chi(uv) = \rho_\chi(u)\rho_\chi(v)$$

for any $u \in U(\mathfrak{g}), \quad v \in Z(\mathfrak{g})$.

Definition $\mathbf{A_h}$ *The algebra* $W_h(\mathfrak{b}_-)$ *is called the Whittaker model of* $Z_h^{s_\pi}(\mathfrak{g})$.

Next we equip $U_h^{s_\pi}(\mathfrak{b}_-)$ with a structure of a left $U_h^{s_\pi}(\mathfrak{n}_+)$ module in such a way that $W_h(\mathfrak{b}_-)$ is identified with the space of invariants with respect to this action. Following Lemma A in the classical case we define this action by

$$x \cdot v = [x, v]^{\chi_h^{s_\pi}}, \tag{2.3}$$

where $v \in U_h^{s_\pi}(\mathfrak{b}_-)$ and $x \in U_h^{s_\pi}(\mathfrak{n}_+)$.

Consider the space $U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}$ of $U_h^{s_\pi}(\mathfrak{n}_+)$ invariants in $U_h^{s_\pi}(\mathfrak{b}_-)$ with respect to this action. Clearly, $W_h(\mathfrak{b}_-) \subseteq U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}$.

Theorem $\mathbf{B_h}$ *Suppose that* $\chi_h^{s_\pi}(e_i) \neq 0$ *(mod h) for* $i = 1, \ldots l$. *Then the space of* $U_h^{s_\pi}(\mathfrak{n}_+)$ *invariants in* $U_h^{s_\pi}(\mathfrak{b}_-)$ *with respect to the action (2.3) is isomorphic to* $W_h(\mathfrak{b}_-)$, *i.e.*

$$U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)} \cong W_h(\mathfrak{b}_-). \tag{2.4}$$

The proof of this theorem, as well as proofs of many statements in this paper, is based on the following Lemma.

Lemma 2.1 *Let* V *be a complete* $\mathbb{C}[[h]]$ *module,* $A, B \subset V$ *two closed subspaces. Denote by* $p : V \rightarrow V/hV$ *the canonical projection. Suppose that* $B \subseteq A$, $p(A) = p(B)$ *and for any* $a \in A, \ b \in B$ *such that* $a - b = hc, \ c \in V$ *we have* $c \in A$. *Then* $A = B$.

Proof. Let $a \in A$. Since $p(A) = p(B)$ one can find an element $b_0 \in B$ such that $a - b_0 = ha_1, \ a_1 \in A$. Applying the same procedure to a_1 one can find elements $b_1 \in B, \ a_2 \in A$ such that $a_1 - b_1 = ha_2$, i.e. $a - b_0 - hb_1 = 0 \ (\text{mod } h^2)$. We can continue this process. Finally we obtain an infinite sequence of elements $b_i \in B$ such that $a - \sum_{i=0}^p h^p b_p = 0 \ (\text{mod } h^{p+1})$. Since the subspace A is closed in the h–adic topology the series $\sum_{i=0}^\infty h^p b_p \in B$ converges to a. Therefore $a \in B$. This completes the proof.

Proof of Theorem B_h. Let $p : U_h^{s_\pi}(\mathfrak{g}) \to U_h^{s_\pi}(\mathfrak{g})/hU_h^{s_\pi}(\mathfrak{g}) = U(\mathfrak{g})$ be the canonical projection. Note that $p(U_h^{s_\pi}(\mathfrak{n}_+)) = U(\mathfrak{n}_+)$, $p(U_h^{s_\pi}(\mathfrak{b}_-)) = U(\mathfrak{b}_-)$ and for every $x \in U_h^{s_\pi}(\mathfrak{n}_+)$ $\chi_h^{s_\pi}(x)$ (mod h) $= \chi(p(x))$ for some non–singular character $\chi : U(\mathfrak{n}_+) \to \mathbb{C}$. Therefore $p(\rho_{\chi_h^{s_\pi}}(x)) = \rho_\chi(p(x))$ for every $x \in U_h^{s_\pi}(\mathfrak{g})$, and hence by Theorem A_q $p(W_h(\mathfrak{b}_-)) = W(\mathfrak{b}_-)$. Using Lemma A and the definition of action (2.3) we also obtain that $p(U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}) = U(\mathfrak{b}_-)^{N_+} = W(\mathfrak{b}_-)$.

Now Theorem $\mathrm{B_h}$ follows immediately from Lemma 2.1 applied to $V = U_h^{s_\pi}(\mathfrak{g})$, $A = U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}$, $B = W_h(\mathfrak{b}_-)$.

2.4 Quantum deformations of Whittaker modules

In this section we define quantum deformations of Whittaker modules. The construction of these modules is similar to that for Lie algebras (see Section 1.3).

Fix a Coxeter element $s_\pi \in W$. Let V_h be a $U_h^{s_\pi}(\mathfrak{g})$ module, which is also free as a $\mathbb{C}[[h]]$ module. The action is denoted by uv for $u \in U_h^{s_\pi}(\mathfrak{g})$, $v \in V_h$. Let $\chi_h^{s_\pi} : U_h^{s_\pi}(\mathfrak{n}_+) \to \mathbb{C}[[h]]$ be a non–singular character of $U_h^{s_\pi}(\mathfrak{n}_+)$ (see Proposition 2.2). We shall assume that $\chi_h^{s_\pi}(e_i) \neq 0$ mod h, $i = 1, \ldots l$. We call a vector $w_h \in V_h$ a Whittaker vector (with respect to $\chi_h^{s_\pi}$) if

$$xw_h = \chi_h^{s_\pi}(x)w_h$$

for all $x \in U_h^{s_\pi}(\mathfrak{n}_+)$. A Whittaker vector w_h is called a cyclic Whittaker vector (for V_h) if $U_h^{s_\pi}(\mathfrak{g})w_h = V_h$. A $U_h^{s_\pi}(\mathfrak{g})$ module V_h is called a Whittaker module if it contains a cyclic Whittaker vector.

Remark 2.4 *Note that in this case $V = V_h/hV_h$ is naturally a Whittaker module for $U(\mathfrak{g})$, $w = w_h$ mod $h \in V$ being a cyclic Whittaker vector for V with respect to the non–singular character χ of $U(\mathfrak{n}_+)$ defined by $\chi(e_i$ mod $h) = \chi_h^{s_\pi}(e_i)$ mod h.*

If V_h is any $U_h^{s_\pi}(\mathfrak{g})$ module we let $U_{h,V}^{s_\pi}(\mathfrak{g})$ be the annihilator of V_h. Then $U_{h,V}^{s_\pi}(\mathfrak{g})$ defines a central ideal $Z_{h,V}^{s_\pi}(\mathfrak{g})$ by putting

$$Z_{h,V}^{s_\pi}(\mathfrak{g}) = Z_h^{s_\pi}(\mathfrak{g}) \cap U_{h,V}^{s_\pi}(\mathfrak{g}). \tag{2.1}$$

Now assume that V_h is a Whittaker module for $U_h^{s_\pi}(\mathfrak{g})$ and $w_h \in V_h$ is a cyclic Whittaker vector. Let $U_{h,w}^{s_\pi}(\mathfrak{g}) \subseteq U_h^{s_\pi}(\mathfrak{g})$ be the annihilator of w_h. Thus

$U^{s_\pi}_{h,V}(\mathfrak{g}) \subseteq U^{s_\pi}_{h,w}(\mathfrak{g})$, where $U^{s_\pi}_{h,w}(\mathfrak{g})$ is a left ideal and $U^{s_\pi}_{h,V}(\mathfrak{g})$ is a two–sided ideal in $U_h^{s_\pi}(\mathfrak{g})$. One has $V_h = U_h^{s_\pi}(\mathfrak{g})/U^{s_\pi}_{h,w}(\mathfrak{g})$ as $U_h^{s_\pi}(\mathfrak{g})$ modules so that V_h is determined up to equivalence by $U^{s_\pi}_{h,w}(\mathfrak{g})$. Clearly $I_{\chi_h^{s_\pi}} = U_h^{s_\pi}(\mathfrak{g})U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}} \subseteq U^{s_\pi}_{h,w}(\mathfrak{g})$ and $U_h^{s_\pi}(\mathfrak{g})Z^{s_\pi}_{h,V}(\mathfrak{g}) \subseteq U^{s_\pi}_{h,w}(\mathfrak{g})$.

The following theorem, similar to Theorem F for Lie algebras, says that up to equivalence V_h is determined by the central ideal $Z^{s_\pi}_{h,V}(\mathfrak{g})$.

Theorem $\mathbf{F_h}$ *Let V_h be any $U_h^{s_\pi}(\mathfrak{g})$ module which admits a cyclic Whittaker vector w_h and let $U^{s_\pi}_{h,w}(\mathfrak{g})$ be the annihilator of w_h. Then*

$$U^{s_\pi}_{h,w}(\mathfrak{g}) = U_h^{s_\pi}(\mathfrak{g})Z^{s_\pi}_{h,V}(\mathfrak{g}) + I_{\chi_h^{s_\pi}}. \tag{2.2}$$

Proof. Denote by $p : U_h^{s_\pi}(\mathfrak{g}) \to U_h^{s_\pi}(\mathfrak{g})/hU_h^{s_\pi}(\mathfrak{g}) \cong U(\mathfrak{g})$ the canonical projection. Then we have $p(U^{s_\pi}_{h,w}(\mathfrak{g})) = U_w(\mathfrak{g})$, where $U_w(\mathfrak{g})$ is the annihilator of the Whittaker vector $w = w_h$ mod h in the Whittaker module $V = V_h/hV_h$ for $U(\mathfrak{g})$ (see Remark 2.4). On the other hand from Theorem F we have $U_w(\mathfrak{g}) = U(\mathfrak{g})Z_V(\mathfrak{g}) + I_\chi$. But clearly $p(U_h^{s_\pi}(\mathfrak{g})Z^{s_\pi}_{h,V}(\mathfrak{g}) + I_{\chi_h^{s_\pi}}) = U(\mathfrak{g})Z_V(\mathfrak{g}) + I_\chi$. Now the result of the theorem follows immediately from Lemma 2.1 applied to $A = U^{s_\pi}_{h,w}(\mathfrak{g})$ and $B = U_h^{s_\pi}(\mathfrak{g})Z^{s_\pi}_{h,V}(\mathfrak{g}) + I_{\chi_h^{s_\pi}}$.

We also have the following quantum counterpart of Lemma B. We use the notation of Section 2.3. If $X \subseteq U_h^{s_\pi}(\mathfrak{g})$ let $X^{\chi_h^{s_\pi}} = \rho_{\chi_h^{s_\pi}}(X)$. Note that $U^{s_\pi}_{h,w}(\mathfrak{g})$ is stable under the map $u \mapsto \rho_{\chi_h^{s_\pi}}(u)$. We recall also that by Theorem $\mathrm{A_h}$ $\rho_{\chi_h^{s_\pi}}$ induces an algebra isomorphism $Z_h^{s_\pi}(\mathfrak{g}) \to W_h(\mathfrak{b}_-)$, where $W_h(\mathfrak{b}_-) = Z_h^{s_\pi}(\mathfrak{g})^{\chi_h^{s_\pi}}$. Thus if $Z_{h,*}$ is any ideal in $Z_h^{s_\pi}(\mathfrak{g})$ then $W_{h,*}(\mathfrak{b}_-) = Z_{h,*}^{\chi_h^{s_\pi}}$ is an isomorphic ideal in $W_h(\mathfrak{b}_-)$. But $(U_h^{s_\pi}(\mathfrak{g})Z_{h,*})^{\chi_h^{s_\pi}} = U_h^{s_\pi}(\mathfrak{b}_-)W_{h,*}(\mathfrak{b}_-)$ by Remark 2.3. Thus by (2.1) one has the direct sum

$$U_h^{s_\pi}(\mathfrak{g})Z_{h,*} + I_{\chi_h^{s_\pi}} = U_h^{s_\pi}(\mathfrak{b}_-)W_{h,*}(\mathfrak{b}_-) \oplus I_{\chi_h^{s_\pi}}. \tag{2.3}$$

Lemma $\mathbf{B_h}$ *Let $X = \{v \in U_h^{s_\pi}(\mathfrak{b}_-) | (x\cdot v)w_h = 0$ for all $x \in U_h^{s_\pi}(\mathfrak{n}_+)\}$, where $x \cdot v$ is given by (2.3). Then*

$$X = U_h^{s_\pi}(\mathfrak{b}_-)W_{h,V}(\mathfrak{b}_-) + W_h(\mathfrak{b}_-), \tag{2.4}$$

where $W_{h,V}(\mathfrak{b}_-) = Z^{s_\pi}_{h,V}(\mathfrak{g})^{\chi_h^{s_\pi}}$. Furthermore if we denote $U^{s_\pi}_{h,w}(\mathfrak{b}_-) = U^{s_\pi}_{h,w}(\mathfrak{g}) \cap U_h^{s_\pi}(\mathfrak{b}_-)$ then

$$U^{s_\pi}_{h,w}(\mathfrak{b}_-) = U_h^{s_\pi}(\mathfrak{b}_-)W_{h,V}(\mathfrak{b}_-). \tag{2.5}$$

Proof is similar to that of Theorem $\mathrm{F_h}$: one should apply Lemma 2.1 to $A = X$ and $B = U_h^{s_\pi}(\mathfrak{b}_-)W_{h,V}(\mathfrak{b}_-) + W_h(\mathfrak{b}_-)$.

Now one can determine, up to equivalence, the set of all Whittaker modules for $U_h^{s_\pi}(\mathfrak{g})$. They are naturally parametrized by the set of all ideals in the center $Z_h^{s_\pi}(\mathfrak{g})$.

Remark 2.5 *The proofs of Theorems* $\mathrm{G_h}$, $\mathrm{H_h}$, $\mathrm{K_h}$ *and* $\mathrm{L_h}$ *below are based on Theorems* $\mathrm{A_h}$, $\mathrm{F_h}$, *Lemma* $\mathrm{B_h}$ *and completely similar to the proofs of Theorems* G, H, K *and* L *in the classical case (see Section 1.3). We do not reproduce these proofs in this section.*

Theorem $\mathbf{G_h}$ *Let V_h be any Whittaker module for $U_h^{s_\pi}(\mathfrak{g})$. Let $U_{h,V}^{s_\pi}(\mathfrak{g})$ be the annihilator of V_h and let $Z_h^{s_\pi}(\mathfrak{g})$ be the center of $U_h^{s_\pi}(\mathfrak{g})$. Then the correspondence*

$$V_h \mapsto Z_{h,V}^{s_\pi}(\mathfrak{g}), \tag{2.6}$$

where $Z_{h,V}^{s_\pi}(\mathfrak{g}) = U_{h,V}^{s_\pi}(\mathfrak{g}) \cap Z_h^{s_\pi}(\mathfrak{g})$, sets up a bijection between the set of all equivalence classes of Whittaker modules and the set of all ideals in $Z_h^{s_\pi}(\mathfrak{g})$.

Now consider the subalgebra $Z_h^{s_\pi}(\mathfrak{g})U_h^{s_\pi}(\mathfrak{n}_+)$ in $U_h^{s_\pi}(\mathfrak{g})$. Using the arguements applied in the proof of Theorem $\mathrm{A_h}$ it is aesy to show that $Z_h^{s_\pi}(\mathfrak{g})U_h^{s_\pi}(\mathfrak{n}_+) \cong Z_h^{s_\pi}(\mathfrak{g}) \otimes U_h^{s_\pi}(\mathfrak{n}_+)$. Now let $Z_{h,*}$ be any ideal in $Z_h^{s_\pi}(\mathfrak{g})$ and regard $Z_h^{s_\pi}(\mathfrak{g})/Z_{h,*}$ as a $Z_h^{s_\pi}(\mathfrak{g})$ module. Equip $Z_h^{s_\pi}(\mathfrak{g})/Z_{h,*}$ with a structure of $Z_h^{s_\pi}(\mathfrak{g}) \otimes U_h^{s_\pi}(\mathfrak{n}_+)$ module by $u \otimes vy = \chi_h^{s_\pi}(v)uy$, where $u \in Z_h^{s_\pi}(\mathfrak{g})$, $v \in U_h^{s_\pi}(\mathfrak{n}_+)$, $y \in Z_h^{s_\pi}(\mathfrak{g})/Z_{h,*}$. We denote this module by $(Z_h^{s_\pi}(\mathfrak{g})/Z_{h,*})_{\chi_h^{s_\pi}}$

The following result is another way of expressing Theorem $\mathrm{G_h}$.

Theorem $\mathbf{H_h}$ *Let V_h be any $U_h^{s_\pi}(\mathfrak{g})$ module. Then V_h is a Whittaker module if and only if one has an isomorphism*

$$V_h \cong U_h^{s_\pi}(\mathfrak{g}) \otimes_{Z_h^{s_\pi}(\mathfrak{g}) \otimes U_h^{s_\pi}(\mathfrak{n}_+)} (Z_h^{s_\pi}(\mathfrak{g})/Z_{h,*})_{\chi_h^{s_\pi}} \tag{2.7}$$

of $U_h^{s_\pi}(\mathfrak{g})$ modules. Furthermore in such a case the ideal $Z_{h,*}$ is unique and is given by $Z_{h,*} = Z_{h,V}^{s_\pi}(\mathfrak{g})$, where $Z_{h,V}^{s_\pi}(\mathfrak{g})$ is defined by (2.1).

Note that the question of reducibility for $U_h^{s_\pi}(\mathfrak{g})$ modules which are free as $\mathbb{C}[[h]]$ modules does not make any sense since if V_h is such a module then hV_h is a proper subrepresentation. However it is natural to study indecomposable

modules for $U_h^{s_\pi}(\mathfrak{g})$. Below we describe all indecomposable Whittaker modules for $U_h^{s_\pi}(\mathfrak{g})$. First we determine all the Whittaker vectors in a Whittaker module for $U_h^{s_\pi}(\mathfrak{g})$.

Theorem $\mathbf{K_h}$ *Let V_h be any $U_h^{s_\pi}(\mathfrak{g})$ module with a cyclic Whittaker vector $w_h \in V_h$. Then any $v \in V_h$ is a Whittaker vector if and only if v is of the form $v = uw_h$, where $u \in Z_h^{s_\pi}(\mathfrak{g})$. Thus the space of all Whittaker vectors in V_h is a cyclic $Z_h^{s_\pi}(\mathfrak{g})$ module which is isomorphic to $Z_h^{s_\pi}(\mathfrak{g})/Z_{h,V}^{s_\pi}(\mathfrak{g})$.*

If V_h is any $U_h^{s_\pi}(\mathfrak{g})$ module then $\mathrm{End}_{U_h}\, V_h$ denotes the algebra of operators on V_h which commute with the action of $U_h^{s_\pi}(\mathfrak{g})$. If $\pi_V : U_h^{s_\pi}(\mathfrak{g}) \to \mathrm{End}\, V_h$ is the representation defining the $U_h^{s_\pi}(\mathfrak{g})$ module structure on V_h then clearly $\pi_V(Z_h^{s_\pi}(\mathfrak{g})) \subseteq \mathrm{End}_{U_h}\, V_h$. Furthermore it is also clear that $\pi_V(Z_h^{s_\pi}(\mathfrak{g})) \cong Z_h^{s_\pi}(\mathfrak{g})/Z_{h,V}^{s_\pi}(\mathfrak{g})$.

Theorem $\mathbf{L_h}$ *Assume that V_h is a Whittaker module. Then $\mathrm{End}_{U_h}\, V_h = \pi_V(Z_h^{s_\pi}(\mathfrak{g}))$. In particular one has an isomorphism*

$$\mathrm{End}_{U_h}\, V_h \cong Z_h^{s_\pi}(\mathfrak{g})/Z_{h,V}^{s_\pi}(\mathfrak{g}).$$

Note that $\mathrm{End}_{U_h}\, V_h$ is commutative.

Now one can describe all indecomposable Whittaker modules for $U_h^{s_\pi}(\mathfrak{g})$. A homomorphism

$$\xi_h : Z_h^{s_\pi}(\mathfrak{g}) \to \mathbb{C}[[h]]$$

is called a central character. Given a central character ξ_h let $Z_{h,\xi_h}^{s_\pi}(\mathfrak{g}) = \mathrm{Ker}\,\xi_h$ so that $Z_{h,\xi_h}^{s_\pi}(\mathfrak{g})$ is a typical central ideal in $Z_h^{s_\pi}(\mathfrak{g})$.

If V_h is any $U_h^{s_\pi}(\mathfrak{g})$ module one says that V_h admits an infinitesimal character, and ξ_h is its infinitesimal character, if ξ_h is a central character such that $uv = \xi_h(u)v$ for all $u \in Z_h^{s_\pi}(\mathfrak{g}),\ v \in V_h$.

Given a central character ξ_h let $\mathbb{C}[[h]]_{\xi_h,\chi_h^{s_\pi}}$ be the 1-dimensional $Z_h^{s_\pi}(\mathfrak{g}) \otimes U_h^{s_\pi}(\mathfrak{n}_+)$ module defined so that if $u \in Z_h^{s_\pi}(\mathfrak{g}),\ v \in U_h^{s_\pi}(\mathfrak{n}_+),\ y \in \mathbb{C}[[h]]_{\xi_h,\chi_h^{s_\pi}}$ then $u \otimes vy = \xi_h(u)\chi_h^{s_\pi}(v)y$. Also let

$$Y_{\xi_h,\chi_h^{s_\pi}} = U_h^{s_\pi}(\mathfrak{g}) \otimes_{Z_h^{s_\pi}(\mathfrak{g}) \otimes U_h^{s_\pi}(\mathfrak{n}_+)} \mathbb{C}[[h]]_{\xi_h,\chi_h^{s_\pi}}.$$

It is clear that $Y_{\xi_h,\chi_h^{s_\pi}}$ admits an infinitesimal character and ξ_h is that character.

Theorem $\mathbf{M_h}$ *Let V_h be any Whittaker module for $U_h^{s_\pi}(\mathfrak{g})$. Then the following conditions are equivalent:*

(1) V_h is an indecomposable $U_h^{s_\pi}(\mathfrak{g})$ module.

(2) V_h admits an infinitesimal character.

(3) The corresponding ideal given by Theorem $\mathrm{G_h}$ *is a maximal ideal.*

(4) The space of Whittaker vectors in V_h is 1–dimensional (over $\mathbb{C}[[h]]$).

(5) The centralizer $\mathrm{End}_{U_h} V_h$ reduces to $\mathbb{C}[[h]]$.

(6) V_h is isomorphic to $Y_{\xi_h,\chi_h^{s_\pi}}$ for some central character ξ_h.

Proof. The equivalence of (2), (3), (4) and (5) follows from Theorems $\mathrm{K_h}$ and $\mathrm{L_h}$. One gets the equivalence with (7) by Theorem $\mathrm{H_h}$. It remains to relate (2)–(6) with (1).

First we shall prove that (1) implies (4). Suppose that V_h is indecomposable. Then the corresponding Whittaker module $V = V_h/hV_h$ for $U(\mathfrak{g})$ is irreducible, and hence by Theorem M the space of Whittaker vectors of V is one–dimensional. We denote this space by $Wh(V)$. Let w_h be a cyclic Whittaker vector for V_h. Denote $Wh'(V_h) = \mathbb{C}[[h]]w_h$. Then clearly $Wh'(V_h) \bmod \mathrm{h} = Wh(V)$. On the other hand if $Wh(V_h)$ is the space of all Whittaker vectors in V_h then $Wh(V_h) \bmod \mathrm{h} = Wh(V)$. Now from Lemma 2.1 applied to $A = Wh(V_h)$, $B = Wh'(V_h)$ it follows that $Wh(V_h) = Wh'(V_h) = \mathbb{C}[[h]]w_h$.

Now we shall prove that (6) implies (1). Assume that (6) is satisfied. Then by construction the corresponding Whittaker module $V = V_h/hV_h$ for $U(\mathfrak{g})$ is isomorphic to $Y_{\xi,\chi}$ for some central character ξ. Now suppose that V_h is decomposable. Then V must be reducible which is impossible by Theorem M. This completes the proof of Theorem $\mathrm{M_h}$.

2.5 Coxeter realizations of quantum groups and Drinfeld twist

In this section we show that the Coxeter realizations $U_h^{s_\pi}(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ are connected with quantizations of some nonstandard bialgebra structures on $\mathfrak{g}$. At the quantum level changing bialgebra structure corresponds to the so–called Drinfeld twist. We shall consider a particular class of such twists described in the following proposition.

Proposition 2.1 ([2], Proposition 4.2.13) *Let $(A, \mu, \imath, \Delta, \varepsilon, S)$ be a Hopf algebra over a commutative ring. Let $\mathcal{F}$ be an invertible element of $A \otimes A$*

such that

$$\mathcal{F}_{12}(\Delta\otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id\otimes\Delta)(\mathcal{F}),$$
$$(\varepsilon\otimes id)(\mathcal{F}) = (id\otimes\varepsilon)(\mathcal{F}) = 1. \tag{2.1}$$

Then, $v = \mu(id\otimes S)(\mathcal{F})$ *is an invertible element of* A *with*

$$v^{-1} = \mu(S\otimes id)(\mathcal{F}^{-1}).$$

Moreover , if we define $\Delta^{\mathcal{F}} : A \to A\otimes A$ *and* $S^{\mathcal{F}} : A \to A$ *by*

$$\Delta^{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad S^{\mathcal{F}}(a) = vS(a)v^{-1},$$

then $(A, \mu, \imath, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ *is a Hopf algebra denoted by* $A^{\mathcal{F}}$ *and called the twist of* A *by* $\mathcal{F}$*.*

Corollary 2.2 ([2], Corollary 4.2.15) *Suppose that* A *and* $\mathcal{F}$ *as in Proposition 2.1, but assume in addition that* A *is quasitriangular with universal R–matrix* $\mathcal{R}$*. Then* $A^{\mathcal{F}}$ *is quasitriangular with universal R–matrix*

$$\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \tag{2.2}$$

where $\mathcal{F}_{21} = \sigma\mathcal{F}$*.*

Fix a Coxeter element $s_\pi \in W$, $s_\pi = s_{\pi(1)}\dots s_{\pi(l)}$. Consider the twist of the Hopf algebra $U_h(\mathfrak{g})$ by the element

$$\mathcal{F} = exp(-h\sum_{i,j=1}^{l}\frac{n_{ji}}{d_j}Y_i\otimes Y_j) \in U_h(\mathfrak{h})\otimes U_h(\mathfrak{h}), \tag{2.3}$$

where n_{ij} is a solution of the corresponding equation (2.2).

This element satisfies conditions (2.1), and so $U_h(\mathfrak{g})^{\mathcal{F}}$ is a quasitriangular Hopf algebra with the universal R–matrix $\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$, where $\mathcal{R}$ is given by (2.10). We shall explicitly calculate the element $\mathcal{R}^{\mathcal{F}}$. Substituting (2.10) and (2.3) into (2.2) and using (2.9) we obtain

$$\mathcal{R}^{\mathcal{F}} = exp\left[h(\textstyle\sum_{i=1}^{l}(Y_i\otimes H_i) + \sum_{i,j=1}^{l}(-\frac{n_{ij}}{d_i}+\frac{n_{ji}}{d_j})Y_i\otimes Y_j)\right]\times$$
$$\textstyle\prod_\beta exp_{q_\beta^{-1}}[(q-q^{-1})a(\beta)^{-1}X_\beta^{+}e^{hK\beta^\vee}\otimes e^{-hK^*\beta^\vee}X_\beta^{-}],$$

where K is defined by (2.4).

Equip $U_h^{s_\pi}(\mathfrak{g})$ with the comultiplication given by : $\Delta_{s_\pi}(x) = (\psi_{\{n\}}^{-1} \otimes \psi_{\{n\}}^{-1})\Delta_h^{\mathcal{F}}(\psi_{\{n\}}(x))$. Then $U_h^{s_\pi}(\mathfrak{g})$ becomes a quasitriangular Hopf algebra with the universal R–matrix $\mathcal{R}^{s_\pi} = \psi_{\{n\}}^{-1} \otimes \psi_{\{n\}}^{-1}\mathcal{R}^{\mathcal{F}}$. Using equation (2.2) and Lemma 2.3 this R–matrix may be written as follows

$$\begin{array}{l} \mathcal{R}^{s_\pi} = exp\left[h(\sum_{i=1}^l (Y_i \otimes H_i) + \sum_{i=1}^l \frac{1+s_\pi}{1-s_\pi} H_i \otimes Y_i)\right] \times \\ \prod_\beta exp_{q_\beta^{-1}}[(q-q^{-1})a(\beta)^{-1} e_\beta \otimes e^{h\frac{1+s_\pi}{1-s_\pi}\beta^\vee} f_\beta]. \end{array} \quad (2.4)$$

The element $\mathcal{R}^{s_\pi}$ may be also represented in the form

$$\begin{array}{l} \mathcal{R}^{s_\pi} = exp\left[h(\sum_{i=1}^l (Y_i \otimes H_i)\right] \times \\ \prod_\beta exp_{q_\beta^{-1}}[(q-q^{-1})a(\beta)^{-1} e_\beta e^{-h\frac{1+s_\pi}{1-s_\pi}\beta^\vee} \otimes f_\beta] exp\left[h(\sum_{i=1}^l \frac{1+s_\pi}{1-s_\pi} H_i \otimes Y_i)\right]. \end{array} \quad (2.5)$$

The comultiplication Δ_{s_π} is given on generators by

$$\Delta_{s_\pi}(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_{s_\pi}(e_i) = e_i \otimes e^{hd_i \frac{2}{1-s_\pi} H_i} + 1 \otimes e_i,$$

$$\Delta_{s_\pi}(f_i) = f_i \otimes e^{-hd_i \frac{1+s_\pi}{1-s_\pi} H_i} + e^{-hd_i H_i} \otimes f_i.$$

Note that the Hopf algebra $U_h^{s_\pi}(\mathfrak{g})$ is a quantization of the bialgebra structure on $\mathfrak{g}$ defined by the cocycle

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) 2r_+^{s_\pi}, \quad r_+^{s_\pi} \in \mathfrak{g} \otimes \mathfrak{g}, \quad (2.6)$$

where $r_+^{s_\pi} = r_+ + \frac{1}{2}\sum_{i=1}^l \frac{1+s_\pi}{1-s_\pi} H_i \otimes Y_i$, and r_+ is given by (2.3).

We shall also need the following property of the antipode S^{s_π} of $U_h^{s_\pi}(\mathfrak{g})$.

Proposition 2.3 *The square of the antipode S^{s_π} is an inner automorphism of $U_h^{s_\pi}(\mathfrak{g})$ given by*

$$(S^{s_\pi})^2(x) = e^{2h\rho^\vee} x e^{-2h\rho^\vee},$$

where $\rho^\vee = \sum_{i=1}^l Y_i$.

Proof. First observe that by Proposition 2.1 the antipode of $U_h^{s_\pi}(\mathfrak{g})$ has the form: $S^{s_\pi}(x) = \psi_{\{n\}}^{-1}(vS_h(\psi_{\{n\}}(x))v^{-1})$, where

$$v = exp(h\sum_{i,j=1}^{l}\frac{n_{ji}}{d_j}Y_iY_j).$$

Therefore $(S^{s_\pi})^2(x) = \psi_{\{n\}}^{-1}(vS_h(v^{-1})S_h^2(\psi_{\{n\}}(x))S_h(v)v^{-1})$. Note that $S_h(v) = v$, and hence $(S^{s_\pi})^2(x) = \psi_{\{n\}}^{-1}(S_h^2(\psi_{\{n\}}(x)))$.

Finally observe that from explicit formulas for the antipode of $U_h(\mathfrak{g})$ it follows that $S_h^2(x) = e^{2h\rho^\vee}xe^{-2h\rho^\vee}$. This completes the proof.

In conclusion we note that using Corollary 2.3 and the isomorphism $\psi_{\{n\}}$ one can define finite–dimensional representations of $U_h^{s_\pi}(\mathfrak{g})$.

2.6 Quantum deformation of the Toda lattice

Recall that one of the main applications of the algebra $W(\mathfrak{b}_-)$ is the quantum Toda lattice [12]. Let $\overline{\chi} : \mathfrak{n}_- \to \mathbb{C}$ be a non–singular character of the opposite nilpotent subalgebra $\mathfrak{n}_-$. We denote the character of N_- corresponding to $\overline{\chi}$ by the same letter. The algebra $U(\mathfrak{b}_-)$ naturally acts by differential operators in the space $C^\infty(\mathbb{C}_{\overline{\chi}}\otimes_{N_-} B_-)$. This space may be identified with $C^\infty(H)$. Let $D_1, \ldots, D_l$ be the differential operators on $C^\infty(H)$ which correspond to the elements $I_1^\chi, \ldots, I_l^\chi \in W(\mathfrak{b}_-)$. Denote by φ the operator of multiplication in $C^\infty(H)$ by the function $\varphi(e^h) = e^{\rho(h)}$, where $h \in \mathfrak{h}$. The operators $M_i = \varphi D_i \varphi^{-1}, i = 1, \ldots l$ are called the quantum Toda Hamiltonians. Clearly, they commute with each other.

In particular if I is the quadratic Casimir element then the corresponding operator M is the well–known second–order differential operator:

$$M = \sum_{i=1}^{l}\partial_i^2 + \sum_{i=1}^{l}\chi(X_{\alpha_i})\overline{\chi}(X_{-\alpha_i})e^{-\alpha_i(h)} + (\rho,\rho),$$

where $\partial_i = \frac{\partial}{\partial y_i}$, and $y_i,\ i = 1, \ldots l$ is an ortonormal basis of $\mathfrak{h}$.

Using the algebra $W_h(\mathfrak{b}_-)$ we shall construct quantum group analogues of the Toda Hamiltonians. A slightly different approach has been recently proposed in [6].

Denote by A the space of linear functions on $\mathbb{C}[[h]]_{\overline{\chi}_h^{s_\pi}} \otimes_{U_h^{s_\pi}(\mathfrak{n}_-)} U_h^{s_\pi}(\mathfrak{b}_-)$, where $\mathbb{C}[[h]]_{\overline{\chi}_h^{s_\pi}}$ is the one–dimensional $U_h^{s_\pi}(\mathfrak{n}_-)$ module defined by $\overline{\chi}_h^{s_\pi}$. Note

that $\mathbb{C}[[h]]_{\overline{\chi}_h^{s_\pi}} \otimes_{U_h^{s_\pi}(\mathfrak{n}_-)} U_h^{s_\pi}(\mathfrak{b}_-) \cong U_h^{s_\pi}(\mathfrak{h})$ as a linear space. Therefore $A = U_h^{s_\pi}(\mathfrak{h})^*$. The algebra $U_h^{s_\pi}(\mathfrak{b}_-)$ naturally acts on $\mathbb{C}[[h]]_{\overline{\chi}_h^{s_\pi}} \otimes_{U_h^{s_\pi}(\mathfrak{n}_-)} U_h^{s_\pi}(\mathfrak{b}_-)$ by multiplications from the right. This action induces an $U_h^{s_\pi}(\mathfrak{b}_-)$–action in the space A. We denote this action by L, $L : U_h^{s_\pi}(\mathfrak{b}_-) \rightarrow \mathrm{End}A$. Clearly, this action generates an action of the algebra $W_h(\mathfrak{b}_-)$ on A.

To construct deformed Toda Hamiltonians we shall use certain elements in $W_h(\mathfrak{b}_-)$. These elements may be described as follows. Let $\mu : U_h^{s_\pi}(\mathfrak{g}) \rightarrow \mathbb{C}[[h]]$ be a map such that $\mu(uv) = \mu(vu)$. By Proposition 2.3 $(S^{s_\pi})^2(x) = e^{2h\rho^\vee} x e^{-2h\rho^\vee}$. Hence from Remark 1 in [5] (see also [7]) it follows that $(id \otimes \mu)(\mathcal{R}_{21}^{s_\pi}\mathcal{R}^{s_\pi}(1 \otimes e^{2h\rho^\vee}))$, where $\mathcal{R}_{21}^{s_\pi} = \sigma\mathcal{R}^{s_\pi}$, is a central element. In particular, for any finite–dimensional $\mathfrak{g}$–module V the element

$$C_V = (id \otimes tr_V)(\mathcal{R}_{21}^{s_\pi}\mathcal{R}^{s_\pi}(1 \otimes e^{2h\rho^\vee})), \tag{2.1}$$

where tr_V is the trace in $V[[h]]$, is central in $U_h^{s_\pi}(\mathfrak{g})$.

Using formulas (2.4) and (2.5) we can easily compute elements $\rho_{\chi_h^{s_\pi}}(C_V) \in W_h(\mathfrak{b}_-)$. For every finite–dimensional $\mathfrak{g}$–module V we have

$$\rho_{\chi_h^{s_\pi}}(C_V) = (id \otimes tr_V)(e^{t_0} \textstyle\prod_\beta exp_{q_\beta^{-1}}[(q - q^{-1})a(\beta)^{-1} f_\beta \otimes e_\beta e^{-h\frac{1+s_\pi}{1-s_\pi}\beta^\vee}] \times$$
$$e^{t_0} \textstyle\prod_\beta exp_{q_\beta^{-1}}[(q - q^{-1})a(\beta)^{-1} \chi_h^{s_\pi}(e_\beta) \otimes e^{h\frac{1+s_\pi}{1-s_\pi}\beta^\vee} f_\beta](1 \otimes e^{2h\rho^\vee})), \tag{2.2}$$

where $t_0 = h\sum_{i=1}^l (Y_i \otimes H_i)$.

We denote by $W_h^{Rep}(\mathfrak{b}_-)$ the subalgebra in $W_h(\mathfrak{b}_-)$ generated by the elements $\rho_{\chi_h^{s_\pi}}(C_V)$, where V runs through all finite–dimensional representations of $\mathfrak{g}$. Note that for every finite–dimensional $\mathfrak{g}$–module V $\rho_{\chi_h^{s_\pi}}(C_V)$ is a polynomial in noncommutative elements $f_i,\ e^{hx},\ x \in \mathfrak{h}$.

Now we shall realize elements of $W_h^{Rep}(\mathfrak{b}_-)$ as difference operators. Let $H_h \in U_h^{s_\pi}(\mathfrak{h})$ be the subgroup generated by elements $e^{hx},\ x \in \mathfrak{h}$. A difference operator on A is an operator T of the form $T = \sum f_i T_{x_i}$ (a finite sum), where $f_i \in A$, and for every $y \in H_h$ $T_x f(y) = (ye^{hx}),\ x \in \mathfrak{h}$.

Proposition 2.1 ([6], Proposition 3.2) *For any $Y \in U_h^{s_\pi}(\mathfrak{b}_-)$, which is a polynomial in noncommutative elements $f_i,\ e^{hx},\ x \in \mathfrak{h}$, the operator $L(Y)$ is a difference operator on A. In particular, the operators $L(I),\ I \in W_h^{Rep}(\mathfrak{b}_-)$ are mutually commuting difference operators on A.*

Proof. It suffices to verify that $L(f_i)$ are difference operators on H_h. Indeed,

$$L(f_i)f(e^{hx}) = f(e^{hx}f_i) = e^{-h\alpha_i(x)}f(f_ie^{hx}) = \overline{\chi}_h^{s_\pi}(f_i)e^{-h\alpha_i(x)}f(e^{hx}).$$

This completes the proof.

Let $\jmath : H_h \to U_h^{s_\pi}(\mathfrak{h})$ be the canonical embedding. Denote $A_h = \jmath^*(A)$. Let T be a difference operator on A. Then one can define a difference operator $\jmath^*(T)$ on the space A_h by $\jmath^*(T)f(y) = T(\jmath(y))$.

Let $D_i^h = \jmath^*(L(\rho_{\chi_h^{s_\pi}}(C_{V_i})))$, where $V_i,\ i = 1, \dots l$ are the fundamental representations of $\mathfrak{g}$. Denote by φ_h the operator of multiplication in A_h by the function $\varphi_h(e^{hx}) = e^{h\rho(x)}$, where $x \in \mathfrak{h}$. The operators $M_i^h = \varphi_h D_i^h \varphi_h^{-1}, i = 1, \dots l$ are called the quantum deformed Toda Hamiltonians.

From now on we suppose that $\pi = id$ and that the ordering of positive roots Δ_+ is fixed as in Proposition 2.5. We denote $s_{id} = s$. Now using formula (2.2) we outline computation of the operators M_i^h. This computation is simplified by the following lemma.

Lemma 2.2 ([6], Lemma 5.2) *Let $X = f_{\gamma_1}...f_{\gamma_n}$. If the roots $\gamma_1, ..., \gamma_n$ are not all simple then $L(X) = 0$. Otherwise, if $\gamma_i = \alpha_{k_i}$, then*

$$\jmath^*(L(X))f(e^{hy}) = e^{-h(\sum \alpha_{k_i},y)}f(e^{hy})\prod_i \overline{\chi}_h^s(f_{k_i})$$

Proof follows immediately from Proposition 2.5 and the arguments used in the proof of Proposition 2.1.

Using this lemma we obtain that if β is not a simple root then the term in (2.2) containing root vector f_β gives a trivial contribution to the operators $L(\rho_{\chi_h^s}(C_{V_i}))$. Note also that by Proposition 2.5 $\chi_h^s(e_\beta) = 0$ if β is not a simple root. Therefore from formula (2.2) we have

$$\begin{aligned} &L(\rho_{\chi_h^s}(C_{V_i})) = \\ &L(id \otimes tr_V)(e^{t_0}\textstyle\prod_i exp_{q^{-2d_i}}[(q_i - q_i^{-1})f_i \otimes e_i e^{-hd_i\frac{1+s}{1-s}H_i}]\times \qquad (2.3) \\ &e^{t_0}\textstyle\prod_i exp_{q^{-2d_i}}[(q_i - q_i^{-1})\chi_h^s(e_i) \otimes e^{hd_i\frac{1+s}{1-s}H_i}f_i](1 \otimes e^{2h\rho^\vee})). \end{aligned}$$

In particular, let $\mathfrak{g} = sl(n)$, $V_1 = V$ the fundamental representation of $sl(n)$. Then direct calculation gives

$$M_1f(e^{hy}) = \left(\sum_{j=1}^n T_j^2 - (q-q^{-1})^2\sum_{i=1}^{n-1}\chi_h^s(e_i)\overline{\chi}_h^s(f_i)e^{-h(y,\alpha_i)}T_{i+1}T_i\right)f(e^{hy}),$$

where $T_i = T_{\omega_i}$, ω_i are the weights of V. The last expression coincides with formula (5.7) obtained in [6].

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